AN ANYON PRIMER

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ABSTRACT

In this set of lectures, we give a pedagogical introduction to the subject of anyons. We discuss 1) basic concepts in anyon physics, 2) quantum mechanics of two anyon systems, 3) statistical mechanics of many anyon systems, 4) mean field approach to many anyon systems and anyon superconductivity , 5) anyons in field theory and 6) anyons in the Fractional Quantum Hall Effect (FQHE).

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This set of lectures is aimed at an audience who may be hearing the term 'anyon' for the first time. We shall start by explaining what the term 'anyon' means and why they are interesting. Just as fermions are spin 1/2, 3/2,...., particles obeying Fermi-Dirac statistics and bosons are spin 0,1,....,particles obeying Bose-Einstein statistics, 'anyons' are particles with 'any' spin obeying 'any' statistics. One may wonder why such particles have not been seen until now. After all, had they really existed in nature, they would have been just as familiar as the usual fermions and bosons. The explanation is quite simple. As will be seen in the course of the lectures, even theoretically, anyons can only exist in two space dimensions, whereas the real world is three dimensional. This naturally leads to the next question, 'why bother to study them at all ?'. The answer is that there do exist phenomena in our three dimensional world that are planar - systems where motion in the third dimension is essentially frozen. Anyons are relevant in the explanation of such phenomena. Besides, the study of anyons has led to a considerable improvement in our theoretical understanding of concepts like quantum statistics.

The theoretical possibility of anyons was put forward as early as 1977 [1]. However, anyons shot into prominence and became a major field of research only in the last few years. One reason for this upsurge of interest was the discovery that the experimentally observed FQHE [2] had a natural explanation [3] [4] in terms of anyons. An even more dramatic rise in its popularity occured when it was discovered that a gas of anyons superconducts [5] [6], when it is coupled to electromagnetism. In fact, for a while, 'anyon superconductivity' was one of the top "candidate" theories to explain high T_c superconductors [7]. Now, due to lack of experimental confirmation, the theory is no longer a 'hot' candidate, but interest in the field of anyons remains as high as ever. The rest of these lectures will involve a more detailed elaboration on the theme of anyons.

We shall first start in Sec.(1) with basic notions of spin and statistics and understand why anyons can only exist in two spatial dimensions [1] [8]. Then we shall study a simple physical model of an anyon that incorporates fractional spin and statistics [9]. In Sec.(2), using this model, we shall solve some simple two anyon quantum mechanics problems, and see that the anyon energy eigenvalues actually interpolate between fermionic and bosonic eigenvalues. We shall also discover that even non-interacting two anyon states are not

simple products of single anyon states [1]. This is the crux of the problem in handling many anyon systems. Here, we shall concentrate on two approximations in which the many anyon problem has been tackled. The quantum statistical mechanics of a many anyon system has been studied [10] via the virial expansion of the equation of state, which is valid in the high temperature, low density limit. In Sec.(3), we shall first briefly review the classical and quantum cluster expansions and the derivation of the virial coefficients in terms of the cluster integrals. Then, using the results of the two anyon problems discussed in Sec.(2), we shall derive the second virial coefficient of the anyon gas [11]. The other approximation in which the many anyon system has been studied is the mean field approach which is valid in the high density, low temperature limit. In this approximation, every anyon sees an 'average' or 'mean' field due to the presence of all the other anyons. Thus, the many body problem is reduced to the problem of a single particle moving in an 'average' potential. It is in this mean field approach that anyon superconductivity has been established [5] [6]. In Sec.(4), we shall study the mean field approach and derive anyon superconductivity in a heuristic way.

Finally, we shall briefly touch upon two slightly more advanced topics, just to give a flavour as to why the study of anyons form such an interesting and relevant field of research today. The more formal topic deals with the formulation of a field theory of anyons. In Sec.(5), we shall introduce anyons in a field theory formalism using a Chern-Simons construction [10] and study an explicit example [12] of a Lagrangian field theory whose topological excitations are anyons. The second topic deals with the more physical question of applicability of anyon physics to condensed matter systems. In Sec.(6), we shall briefly indicate how anyons arise in the FQHE, which is a system of two dimensional electrons at low temperatures and in strong magnetic fields and show how the idea of statistics transmutation is used in novel explanations [13] [14] of the effect. We wish to emphasise here that realistic anyons do not exist in vacuum as has been assumed in the earlier sections, but actually arise as quasiparticles in a real medium.

1. Basic Concepts in Anyon Physics

Let us start with spin in the familiar three dimensional world. We know that spin is an intrinsic angular momentum quantum number that labels different particles. The three spatial components of the spin obey the commutation relations given by

$$[S_i, S_j] = i\epsilon_{ijk} S_k. \tag{1.1}$$

We shall show that these commutation relations constrain $\mathbf{S}$ to be either integer or half-integer. Let $|s, m\rangle$ be the state with $S^2|s, m\rangle = s(s+1)|s, m\rangle$ and $S_3|s, m\rangle = m|s, m\rangle$. By applying the raising operator S^+, we may create the state

$$S^+|s, m\rangle = [s(s+1) - m(m+1)]^{1/2}|s, m+1\rangle = |s, m'\rangle. \tag{1.2}$$

Requiring this state to have positive norm leads to

$$s(s+1) - m(m+1) \geq 0 \qquad \forall \qquad m, \tag{1.3}$$

which in turn leads to

$$m \leq s \qquad \forall \qquad m. \tag{1.4}$$

Thus, it is clear that for some value of $m' = m + \text{integer}$, $m' > s$ unless $s = m'$ - *i.e.*,

$$s - m = \text{integer}. \tag{1.5}$$

Similarly, by insisting that $S^- = |s, m\rangle$ have positive norm, we get

$$s(s+1) - m(m-1) \geq 0 \qquad \forall \qquad m, \tag{1.6}$$

which in turn implies that

$$m \geq -s \qquad \forall \qquad m. \tag{1.7}$$

Once again, we construct the states, $S^-|s, m\rangle$, $(S^-)^2|s, m\rangle$,... and to avoid $m < -s$, we

have to set

$$m - (-s) = \text{integer}. \tag{1.8}$$

Adding equations (1.5) and (1.8), we get

$$2s = \text{integer} \Longrightarrow s = \frac{\text{integer}}{2}. \tag{1.9}$$

Thus, just from the commutation relations, we have proven that particles in 3+1 dimensions have either integral or half-integral spin.

In two spatial dimensions, however, there exists only one axis of rotation (the axis perpendicular to the plane). Hence, here spin only refers to S_3, which has no commutation relations to satisfy. For a given magnitude of S_3, it can only be either positive or negative depending upon the handedness of the rotation in the plane. Since there are no commutation relations to satisfy, there is no constraint on S_3 and hence, we can have 'any' spin in two dimensions. For completeness, note that in one spatial dimension, there is no axis for rotation and hence, no notion of spin.

The term statistics refers to the phase picked up by a wavefunction when two identical particles are exchanged. However, this definition is slightly ambiguous. Does statistics refer to the phase picked up by the wavefunction when all the quantum numbers of the two particles are exchanged (*i.e.*, under permutation of the particles) or the actual phase that arises when two particles are adiabatically transported giving rise to the exchange? In three dimension, both these definitions are equivalent, but not so in two dimensions. We shall concentrate on the second definition which is more crucial to physics and return to the first definition later.

Let us first consider the statistics of two identical particles moving in three space dimensions [1] [8]. The configuration space is given by the set of pairs of position vectors $(\mathbf{r}_1, \mathbf{r}_2)$. The indistinguishability of identical particles implies the identification $(\mathbf{r}_1, \mathbf{r}_2) \sim (\mathbf{r}_2, \mathbf{r}_1)$ - *i.e.*, we cannot say whether the first particle is at $\mathbf{r}_1$ and the second particle at $\mathbf{r}_2$ or the other way around. We shall also impose the hard-core constraint, $\mathbf{r}_1 \neq \mathbf{r}_2$, to prevent intersecting trajectories so that we can determine whether or not two particles have

been exchanged. However, as we shall see later, this constraint is unnecessary because for all particles (except bosons), there is an automatic angular momentum barrier, preventing two particles from intersecting, whereas for bosons, we need not know whether the particles have been exchanged or not, since the phase is one anyway. For convenience in constructing the configuration space, let us define the centre of mass (CM) and relative coordinates - $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. In terms of these coordinates, the configuration space is $(\mathbf{R}, \mathbf{r})$ with $\mathbf{r} \neq 0$, and with $\mathbf{r}$ being identified with $-\mathbf{r}$. This can be written as

$$R_3 \otimes (\frac{R_3 - \text{origin}}{Z_2}). \tag{1.10}$$

Here R_3 denotes the three dimensional Euclidean space spanned by $\mathbf{R}$. The notation (R_3-origin) for $\mathbf{r}$ implies that the origin $\mathbf{r} = 0$ is being dropped. Z_2 is just the multiplicative group of the two numbers 1 and -1. Hence, dividing (R_3 − origin) by Z_2 implies the identification of every position vector $\mathbf{r}$ in the relative space with its negative $-\mathbf{r}$. To study the phase picked up by the wavefunction of a particle as it moves around the other particle, we need to classify all possible closed paths in the configuration space. Notice that the CM motion just shifts the positions of the two particles together and is independent of any possible phase under exchange. Hence, we are only required to classify closed paths in

$$\frac{(R_3 - \text{origin})}{Z_2} = S. \tag{1.11}$$

Instead of dealing with paths in S, for ease of visualisation, let us construct a simpler configuration space by keeping the magnitude of $\mathbf{r}$ fixed , so that the tip of $\mathbf{r}$ defines the surface of a sphere. Furthermore, the identification of $\mathbf{r}$ with $-\mathbf{r}$ implies that diametrically opposite points on the sphere are identified. Thus, configuration space is the surface of a sphere with opposite points identified as shown in Figs.(1a,1b,1c).

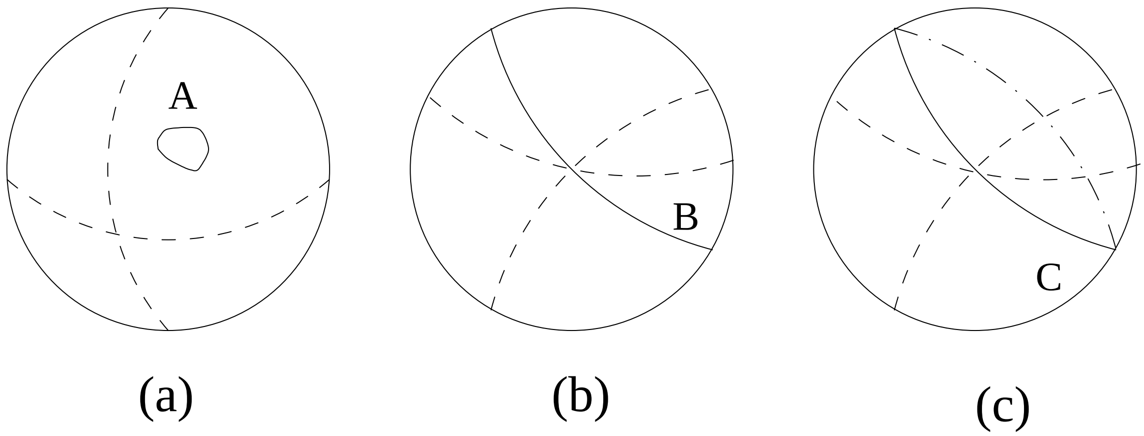


Fig. 1

Now, since we have eliminated coincident points, the wavefunction is non-singular and well-defined at all points in the configuration space. In particular, it is non-singular on the surface of the sphere. Hence, the phase picked up by the wavefunction under an adiabatic exchange of the two particles is also well-defined and does not change under continuous deformations of the path. Let us consider the possible phases of the wavefunction when the motion of the particles is along each of the three paths - A (no exchange), B (single exchange) and C (two exchanges) - depicted in Figs.(1a, 1b, 1c). Path A defines a motion of the particles which does not involve any exchange. It is clearly a closed path on the surface of the sphere and can be continuously shrunk to a point. So this path cannot impart any phase to the wavefunction. Path B, on the other hand, involves the exchange of two particles and goes from a point on the sphere to its diametrically opposite point - again a closed path. Since the two endpoints are fixed, by no continuous process can this path be shrunk to a point. Hence, this path can cause a non-trivial phase in the wavefunction. However, path C which involves two exchanges, forms a closed path on the surface of the sphere, which, by imagining the path to be a (physical!) string looped around an orange (surface of a sphere), can be continuously shrunk to a point. So once again, the wavefunction cannot pick up any phase under two adiabatic exchanges. This leads us to conclude that there are only two classes of closed paths that are possible in this configuration space - single exchange or no exchange. Let η be the phase picked up by

any particle under single exchange. The fact that two exchanges are equal to no exchanges implies that

$$\eta^2 = +1 \Longrightarrow \eta = \pm 1. \tag{1.12}$$

Hence, the only statistics that are possible in three space dimensions are Bose statistics and Fermi statistics.

Why does this argument break down in two space dimensions? Here, configuration space is given by

$$R_2 \otimes \frac{(R_2 - \text{origin})}{Z_2}, \tag{1.13}$$

where R_2 is two dimensional Euclidean space. Just as before, we ignore the CM motion and fix the magnitude of the relative separation, so that the configuration space can be visualised as a circle with diametrically opposite points identified (see Figs.(2a,2b,2c)).

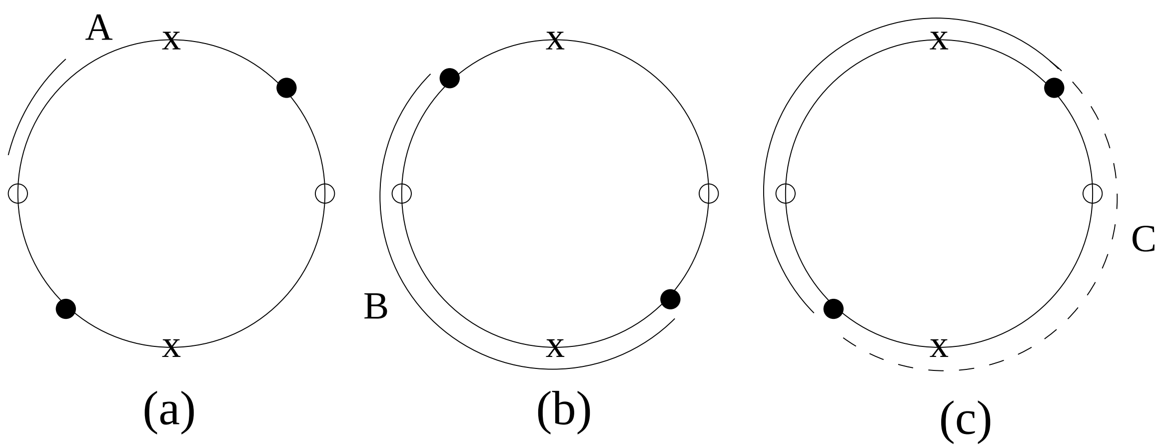


Fig. 2

Here, however, several closed paths are possible. The path A that involves no exchanges can obviously shrunk to a point since it only involves motion along any segment of the circle and back. However, the path B that exchanges the two particles is just as obviously not contractible since the end-points are fixed. But even the path C, where both the dashed and the solid lines are followed in the clockwise direction (or equivalently both in the anti-clockwise direction) cannot be contracted to a point. This is easily understood

by visualising the paths as physical strings looping around a cylinder (a circle in a plane). Thus, if η is the phase under single exchange, η^2 is the phase under two exchanges, η^3 is the phase under three exchanges and so on. All we can say is that since the modulus of the wavefunction remains unchanged under exchange, η is a phase and can be written as $e^{i\theta}$, where θ is called the statistics parameter. This explains why we can have 'any' statistics in two spatial dimensions.

The crux of the distinction between configuration spaces in two and three dimensions is that the removal of the origin in two dimensional space makes the space multiply connected, whereas three dimensional space remains singly connected. Hence, in two dimensions, it is possible to define paths that wind around the origin. This is not possible in three dimensions. Mathematically, this distinction is expressed in terms of the first homotopy group Π_1, which is the group formed by inequivalent paths (paths that are not deformable to one another), passing through a given point in configuration space, with group multiplication being defined as traversing paths in succession and group inverse as traversing a path in the opposite direction. Thus, in two dimensions,

$$\Pi_1(2\,\mathrm{dim.\,config.\,space}) = \Pi_1 \frac{(R_2 - \mathrm{origin})}{Z_2} = \Pi_1(RP_1) = Z \tag{1.14}$$

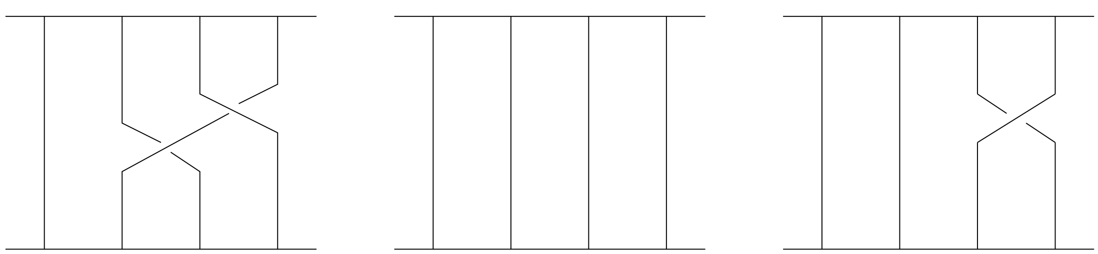
where Z is the group of integers under addition and RP_1 stands for real projective one dimensional space and is just the notation for the circumference of a circle with diametrically opposite points identified. The equality in the above equation stands for isomorphism of the groups so that the homotopy group of configuration space is isomorphic to the group of integers under addition. The different paths are labelled by integer winding numbers, so that the phases developed by the wavefunction are of the form, η^n, with n an integer , which in turn leads to the possibility of 'any' statistics in two dimensions. In contrast, in three dimensions, we have

$$\Pi_1(3\,\mathrm{dim.\,config.\,space}) = \Pi_1 \frac{(R_3 - \mathrm{origin})}{Z_2} = \Pi_1(RP_2) = Z_2. \tag{1.15}$$

Here RP_2 stands for real two dimensional projective space and is the notation for the surface of a sphere with diametrically opposite points identified. Since, Z_2 has only two

elements, there exist only two classes of independent paths and thus, only two possible phases - fermionic or bosonic - in three spatial dimensions.

The distinction between the phase of the wavefunction under exchange of quantum numbers and the phase obtained after adiabatic transport of particles is also now clear. Under the former definition, the phase η^2 after two exchanges is always unity, so that $\eta = \pm 1$, whereas the phase under the latter definition has many more possibilities in two dimensions. Mathematically, the distinction is that the first definition classifies particles under the permutation group P_N, whereas the second definition classifies particles under the braid group B_N. The permutation group (P_N) is the group formed by all possible permutations of N objects with group multiplication defined as successive permutations and group inverse defined as undoing the permutation. It is clear that the square of any permutation is just unity, since permuting two objects twice brings the system back to the original configuration. Thus, particles that transform as representations of P_N can only be fermions or bosons. The braid group B_N, on the other hand, is the group of inequivalent paths (or trajectories) that occur when adiabatically transporting N objects. For example, the trajectories shown in Fig.(3)



(a) (b) (c)

Fig. 3

are elements of B_4, because all of them are possible paths involving four particles. Fig.(3b) represents the identity element where none of the trajectories cross each other. Group multiplication is defined as following one trajectory by another as depicted in Fig.(4)

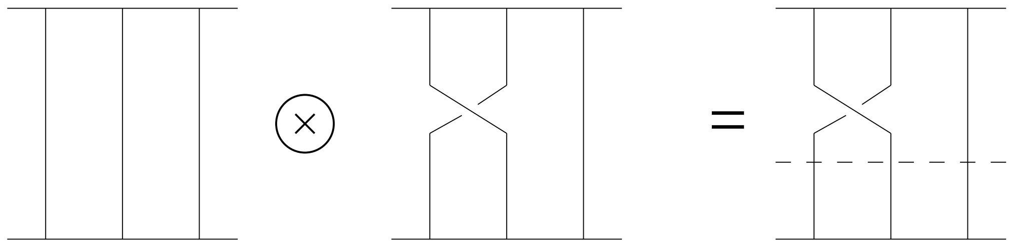


Fig. 4

and group inverse is defined as a reverse crossing, (Fig.(5))

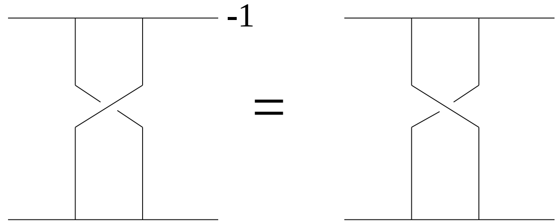


Fig. 5

so that the product of a trajectory and its inverse leads to the identity as shown in Fig.(6).

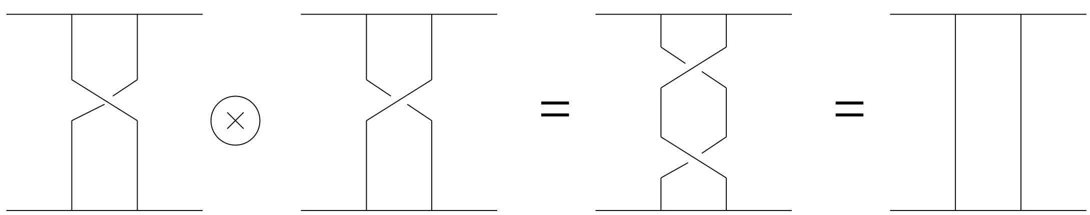


Fig. 6

Here, it is pictorially clear (see Fig.(7)),

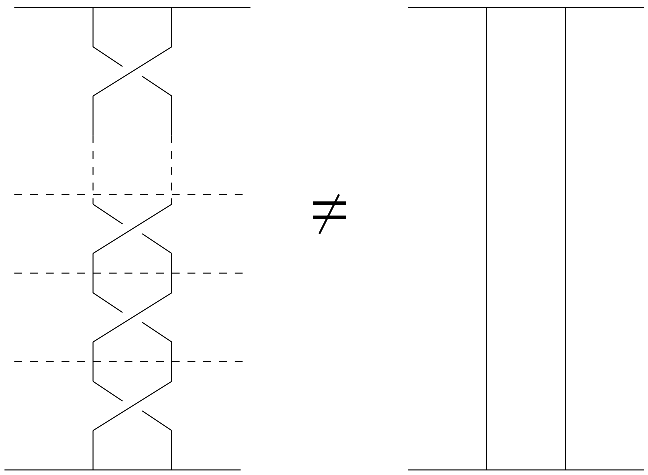


Fig. 7

that the square, or indeed, any power of the trajectory representing the adiabatic exchange of two particles is not 1. Hence, particles that transform as representations of the braid group are allowed to pick up 'any' phase under adiabatic exchange. For completeness, we mention that more abstractly, the braid group B_N is defined as the group whose elements (trajectories) satisfy the following two relations depicted pictorially in Fig.(8)

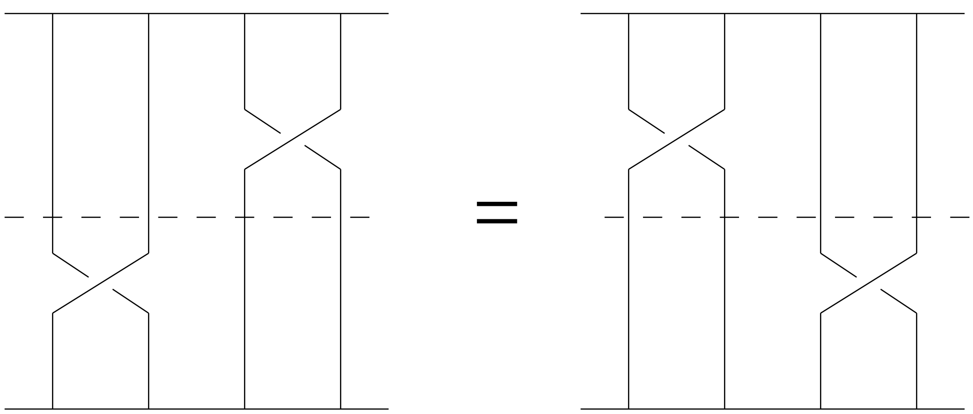


Fig. 8

and Fig.(9).

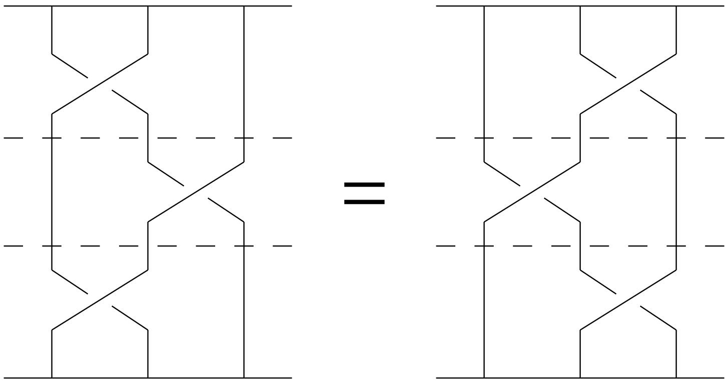


Fig. 9

The second relation is called the Yang-Baxter relation. It is clear that the braid group is a much richer group than the permutation group and leads to a much finer classification. For example, consider the trajectories a) and b) in Fig.(10).

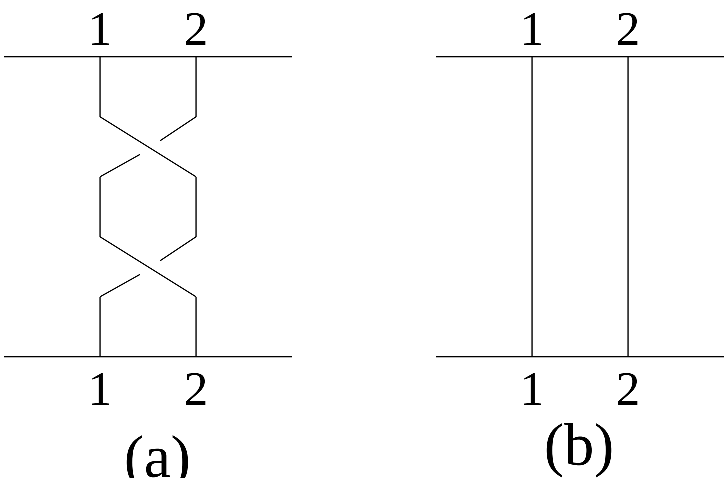


Fig. 10

The permutation group cannot distinguish between these two trajectories. In both cases, there is no permutation and hence the trajectory belongs to the identity representation, whereas the two trajectories are distinct elements of the braid group.

Finally notice that in one spatial dimension, two particles cannot be adiabatically exchanged without passing through each other -*i.e.*, without interacting. Hence any theory can be written in terms of bosons or fermions with appropriate interactions and there is no

real concept of statistics in one dimension.

After all this abstract discussion, let us construct a simple physical model of an anyon [9]. Imagine a spinless particle of charge q orbiting around a thin solenoid along the z-axis, at a distance $\mathbf{r}$ as shown in Fig.(11).

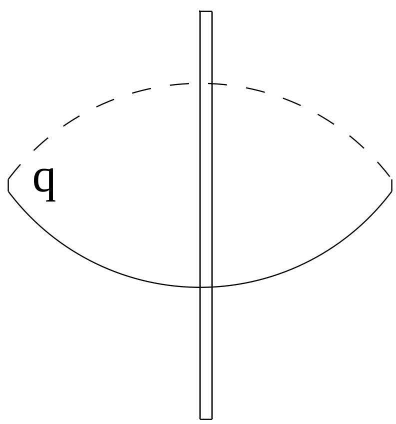


Fig. 11

When there is no current flowing through the solenoid, the orbital angular momentum of the charged particle is quantised as an integer -*i.e.*,

$$l_z = \text{integer}. \tag{1.16}$$

When a current is turned on, the particle feels an electric field that can easily be computed using

$$\int (\nabla \times \mathbf{E}) d^2\mathbf{r} = -\frac{\partial}{\partial t} \int B d^2\mathbf{r} = -\frac{\partial \phi}{\partial t} \tag{1.17}$$

where ϕ is the total flux through the solenoid. Hence,

$$\int \mathbf{E} \cdot d\mathbf{l} = 2\pi |\mathbf{r}| E_\theta = -\dot{\phi} \tag{1.18}$$

leading to

$$\mathbf{E} = -\frac{\dot{\phi}}{2\pi |\mathbf{r}|} (\hat{z} \times \hat{\mathbf{r}}). \tag{1.19}$$

Thus, the angular momentum of the charged particle changes, with the rate of change being

proportional to the torque $\mathbf{r} \times \mathbf{F}$ -*i.e.*,

$$\dot{l}_z = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times q\mathbf{E} = -\frac{q\dot{\phi}}{2\pi} \tag{1.20}$$

Hence,

$$\Delta l_z = -\frac{q\phi}{2\pi} \tag{1.21}$$

is the change in angular momentum due to the flux ϕ through the solenoid. In the limit where the solenoid becomes extremely narrow and the distance between the solenoid and the charged particle is shrunk to zero, the system may be considered as a single composite object - a charged particle-flux tube composite. Furthermore, for a planar system, there can be no extension in the z-direction. Hence, imagine shrinking the solenoid along the z-direction also to a point. The composite object is now pointlike, has fractional angular momentum and perhaps can be identified as a model for an anyon. However, this is too naive a picture. As we shall see later, in an anyon, the charge and the flux that it carries are related. So it is not quite right to think of the anyon as an independent charge orbiting around an independent flux. The charge is actually being switched on at the same time that the flux in the solenoid is being switched on. Hence, Eq.(1.20) needs to be modified to read

$$\dot{l}_z = -\frac{q(t)\dot{\phi}}{2\pi}. \tag{1.22}$$

Moreover, since $q(t) = c\phi(t)$, for some constant c, we get

$$\Delta l_z = -\frac{c\phi^2}{4\pi} = -\frac{q\phi}{4\pi}, \tag{1.23}$$

so that the angular momentum of a charge-flux composite, with charge proportional to the flux, is less than what we originally computed. This is not surprising since the original computation overestimated the charge by keeping it fixed. Hence, our final physical model of an anyon is that of a charge-flux composite with charge q and flux ϕ being proportional to each other and with a spin given by $q\phi/4\pi$. In the next section, we shall see that this model also exhibits fractional statistics and complete its identification as an anyon.

Problems

1. Fermions and bosons are one-dimensional representations of the permutation group P_N and anyons are one-dimensional representations of the braid group B_N. Can one have higher dimensional representations of P_N and B_N? (The answer is yes.) Think about the quantum mechanics and statistical mechanics of particles in these representations and see how far you get.

2. Quantum Mechanics of Two Anyon Systems

In Sec.(1), we constructed a composite object which consisted of a spinless (bosonic) charge orbiting around a (bosonic) flux, and showed that when the charge is proportional to the flux, this object had fractional spin $s = q\phi/4\pi$. To determine its statistics, we need to study the quantum mechanical system of two such objects. The Hamiltonian for the system is given by

$$H = \frac{(\mathbf{p}_1 - q\mathbf{a}_1)^2}{2m} + \frac{(\mathbf{p}_2 - q\mathbf{a}_2)^2}{2m} \tag{2.1}$$

with

$$\begin{aligned} \mathbf{a}_1 &= \frac{\phi}{2\pi}\frac{\hat{z}\times(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^2}, \\ \text{and} \qquad \mathbf{a}_2 &= \frac{\phi}{2\pi}\frac{\hat{z}\times(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_1 - \mathbf{r}_2|^2}, \end{aligned} \tag{2.2}$$

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are the vector potentials at the positions of the composites 1 and 2, due to the fluxes in the composites 2 and 1 respectively. Let us work in the centre of mass (CM) and relative (rel) coordinates - i.e., we define respectively,

$$\mathbf{R} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} \Rightarrow \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \qquad \text{and} \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \Rightarrow \mathbf{p} = \frac{\mathbf{p}_1 - \mathbf{p}_2}{2}. \tag{2.3}$$

In terms of these coordinates, the Hamiltonian can be recast as

$$H = \frac{\mathbf{P}^2}{4m} + \frac{(\mathbf{p} - q\mathbf{a}_{\mathrm{rel}})^2}{m} \tag{2.4}$$

with

$$\mathbf{a}_{\mathrm{rel}} = \frac{\phi}{2\pi}\frac{\hat{z}\times\mathbf{r}}{|\mathbf{r}|^2}. \tag{2.5}$$

Thus, the CM motion, which translates both the particles rigidly and is independent of statistics, is free. The relative motion, on the other hand, which is sensitive to whether the composites are bosons, fermions or anyons, has reduced to the system of a single charged particle of mass $m/2$ orbiting around a flux ϕ at a distance $\mathbf{r}$. Since the composite has been formed of a bosonic charge orbiting around a bosonic flux, the wavefunction of the

two composite system is symmetric under exchange and the boundary condition is given by

$$\psi_{\rm rel}(r, \theta + \pi) = \psi_{\rm rel}(r, \theta) \tag{2.6}$$

where $\psi_{\rm rel}$ is the wavefunction of the relative piece of the Hamiltonian in Eq.(2.4) and $\mathbf{r} = (r, \theta)$ in cylindrical coordinates.

Now, let us perform a (singular) gauge transformation so that

$$\mathbf{a}_{\rm rel} \longrightarrow \mathbf{a}'_{\rm rel} = \mathbf{a}_{\rm rel} - \nabla\Lambda(r, \theta) \tag{2.7}$$

where $\Lambda(r, \theta) = \frac{\phi}{2\pi}\theta$. This gauge transformation is singular because θ is a periodic angular coordinate with period θ and is not single valued. In the primed gauge,

$$a'_{{\rm rel}\theta} = a_{{\rm rel}\theta} - \frac{1}{r}\frac{\partial}{\partial\theta}(\frac{\phi\theta}{2\pi}) = 0 \qquad \text{and} \qquad a'_{{\rm rel}r} = a_{{\rm rel}r} = 0 \tag{2.8}$$

-i.e., the gauge potential completely vanishes. Hence, the Hamiltonian reduces to

$$H = \frac{\mathbf{P}^2}{4m} + \frac{\mathbf{p}^2}{m} \tag{2.9}$$

which is just the Hamiltonian of two free particles. However, in the primed gauge, the wavefunction of the relative Hamiltonian has also changed. It is now given by

$$\psi'_{\rm rel}(r, \theta) = e^{-iq\Lambda}\psi_{\rm rel}(r, \theta) = e^{-i\frac{q\phi}{2\pi}\theta}\psi_{\rm rel}(r, \theta) \tag{2.10}$$

which is no longer symmetric under $\mathbf{r} \to -\mathbf{r}$ since

$$\psi'_{\rm rel}(r, \theta + \pi) = e^{-iq\phi/2}\psi'_{\rm rel}(r, \theta) \equiv e^{-i\alpha}\psi'_{\rm rel}(r, \theta). \tag{2.11}$$

Thus, two interacting bosonic charge-flux composites are equivalent to two free particles whose wavefunctions develop a phase $e^{-iq\phi/2}$ under exchange - i.e., they obey fractional statistics. This completes the identification of charge-flux composites as anyons. Notice that the statistics phase $\alpha = q\phi/2$ is in accordance with the spin of the composite, which we had earlier determined to be $q\phi/4\pi$, so that the generalised spin-statistics theorem which relates the statistics factor α to the spin $j = \alpha/2\pi$, is satisfied.

We have thus shown that anyons can either be considered as free particles with fractional spin obeying fractional statistics, or as interacting charge-flux composites, again with fractional spin, but obeying Bose statistics. The free particle representation is called the anyon gauge, and the interacting particle representation is called the boson gauge.

Let us now solve the quantum mechanical problem of two free anyons. The Hamiltonian in Eq.(2.9) is simple, but the boundary condition in Eq.(2.11) is non-trivial. However, we have just seen that this problem is equivalent to the problem of two interacting charge-flux composites, described by the Hamiltonian in Eq. (2.4) with bosonic boundary conditions. The CM motion is trivial and directly solved to yield the energy eigenvalues and eigen functions as

$$E_{\mathrm{CM}} = \frac{\mathbf{P}^2}{4m} \qquad \text{and} \qquad \psi_{\mathrm{CM}} = e^{i\mathbf{P}\cdot\mathbf{R}}. \tag{2.12}$$

To solve for the relative motion, we work in cylindrical coordinates, wherein the relative part of the Hamiltonian equation takes the form

$$[-\frac{1}{m}(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}) + \frac{1}{mr^2}(i\frac{\partial}{\partial \theta} - \frac{q\phi}{2\pi})^2]\psi_{\mathrm{rel}}(r,\theta) = E_{rel}\psi_{\mathrm{rel}}(r,\theta) \tag{2.13}$$

with the boundary conditions in Eq.(2.6). Since the Hamiltonian is separable in r and θ, the wave function factorises as $\psi_{\mathrm{rel}}(r,\theta) = \mathcal{R}(r)Y_l(\theta)$. Hence, the angular equation reduces to

$$(i\frac{\partial}{\partial \theta} - \frac{q\phi}{2\pi})^2 Y_l(\theta) = \lambda Y_l(\theta) \tag{2.14}$$

whose solution for $Y_l(\theta)$, consistent with the boundary condition is given by

$$Y_l(\theta) = e^{il\theta}, \qquad l = \text{even integer}, \tag{2.15}$$

which, in turn, leads to

$$\lambda = (l + \frac{q\phi}{2\pi})^2, \qquad l = \text{even integer}. \tag{2.16}$$

When the λ eigenvalue is substituted in the radial equation, we get

$$[-\frac{1}{m}(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}) + \frac{1}{mr^2}(l + \frac{q\phi}{2\pi})^2]\mathcal{R}(r) = E_{\mathrm{rel}}\mathcal{R}(r). \tag{2.17}$$

Notice that the net effect of the fluxtubes has been to add a factor $q\phi/2\pi$ to the angular momentum term, thus adding to the centrifugal barrier. This justifies our earlier statement in Sec.(1) that all particles, except bosons, have a centrifugal barrier which prevents

intersecting trajectories. Now, by defining $mE_{\rm rel} = k^2$, we may rewrite the radial equation in the form

$$[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}(l + \frac{q\phi}{2\pi})^2 + k^2]\mathcal{R}(r) = 0, \tag{2.18}$$

which is easily identified as the Bessel equation with the solution

$$\mathcal{R}(r) = J_{|l+\frac{q\phi}{2\pi}|}(kr). \tag{2.19}$$

Hence the relative wavefunction of the composites, in the bosonic gauge, is given by

$$\psi_{\rm rel}(r,\theta) = e^{il\theta} J_{|l+\frac{q\phi}{2\pi}|}(kr), \qquad l = \text{even integer}. \tag{2.20}$$

The wavefunction for two anyons can also be expressed in the anyon gauge by performing a gauge transformation so that

$$\psi'_{\rm rel} = e^{i(l+\frac{q\phi}{2\pi})\theta} J_{|l+\frac{q\phi}{2\pi}|}(kr), \qquad l = \text{even integer}. \tag{2.21}$$

This wavefunction is obviously anyonic, since it picks up a phase $q\phi/2$ under $\theta \longrightarrow \theta + \pi$ (*i.e.*, under exchange of the two particles).

Including the CM motion, the two anyon wavefunction can be written as

$$\psi(\mathbf{R},\mathbf{r}) = \psi_{\rm CM}(\mathbf{R})\psi_{\rm rel}(\mathbf{r}) = e^{i\mathbf{P}\cdot\mathbf{R}} e^{i(l+\frac{q\phi}{2\pi})\theta} J_{|l+\frac{q\phi}{2\pi}|}(kr) \tag{2.22}$$

or equivalently as

$$\psi(\mathbf{R},\mathbf{r}) = e^{iL\Theta} J_L(KR) e^{i(l+\frac{q\phi}{2\pi})\theta} J_{|l+\frac{q\phi}{2\pi}|}(kr). \tag{2.23}$$

In Eq.(2.23), we have expressed the CM motion also in terms of cylindrical quantum numbers - L is the CM angular momentum and $K = 4mE_{\rm CM}$ labels the CM energy. The two particle wavefunction can be recast in terms of the original single particle coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ - *i.e.*, $\psi(\mathbf{R},\mathbf{r}) \longrightarrow \psi(\mathbf{r}_1,\mathbf{r}_2)$. However, unless $\frac{q\phi}{2\pi}$ is either integral or half-integral, the two particle wavefunction cannot be factorised into a product of two suitable one particle wavefunctions — $\psi(\mathbf{r}_1,\mathbf{r}_2) \neq \psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2)$. Since handling Bessel functions is inconvenient, this property will be demonstrated more explicitly in the next example of two anyons in a

harmonic oscillator potential. Furthermore, in principle, we should be able to prove from this example that the energy levels of a two anyon system cannot be obtained as sums of one anyon energy levels. This, again, is easier to see with discrete energy levels and will be explicitly proven in the next example.

The second example that we shall solve explicitly is the problem of two anyons in a harmonic oscillator potential [1] with the Hamiltonian given by

$$H = \frac{\mathbf{p}_1^2}{2m} + \frac{\mathbf{p}_2^2}{2m} + \frac{1}{2}m\omega^2\mathbf{r}_1^2 + \frac{1}{2}m\omega^2\mathbf{r}_2^2. \tag{2.24}$$

As in the case of two free anyons, the problem is separable in the CM and relative coordinates, in terms of which, the Hamiltonian can be rewritten as

$$H = \frac{\mathbf{P}^2}{4m} + \frac{\mathbf{p}^2}{m} + m\omega^2\mathbf{R}^2 + \frac{1}{4}m\omega^2\mathbf{r}^2. \tag{2.25}$$

The CM motion is clearly independent of the statistics of the particles and involves just the usual quantum mechanical problem of a single particle of mass $2m$ in a two dimensional harmonic oscillator potential. The energy levels and wavefunctions are obviously well-known. However, we shall briefly recollect the familiar steps here, just to set the field for the relative motion problem which is sensitive to the statistics of the particles. For the CM motion, we work in the cylindrical (R, Θ) coordinates and write

$$H_{\mathrm{CM}}\psi_{\mathrm{CM}}(R,\Theta) = [-\frac{1}{4m}(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R}) - \frac{1}{4mR^2}\frac{\partial^2}{\partial\Theta^2} + m\omega^2R^2]\psi_{\mathrm{CM}}(R,\Theta). \tag{2.26}$$

The wavefunction factorises in R and Θ and can be written as

$$\psi_{\mathrm{CM}}(R,\Theta) = e^{iL\Theta}\mathcal{R}_{\mathrm{CM}}(R) \tag{2.27}$$

with L being an integer for a single valued wavefunction. Defining $K^2 = 4mE_{\mathrm{CM}}$, the radial eigenvalue equation becomes

$$[\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} - \frac{L^2}{R^2} - 4m^2\omega^2R^2 + K^2]\mathcal{R}_{\mathrm{CM}}(R) = 0, \tag{2.28}$$

with a solution of the form

$$\mathcal{R}_{\rm CM}(R) = e^{-m\omega R^2}\sum_{n=o}^{\infty} a_n R^{n+s}. \tag{2.29}$$

By substituting this series solution in Eq.(2.28), we find that $s = |L|$ and that

$$\frac{a_{n+2}}{a_n} = \frac{K^2 - 4m\omega(n+|L|+1)}{(n+2)^2 + 2|L|(n+2)}, \tag{2.30}$$

so that $n = \text{even integer}$. Requiring the series to terminate leads to

$$E_{\rm CM} = \omega(n + |L| + 1) \tag{2.31}$$

which is the well known answer, since $n + |L| = p$ is an integer, and the modulus factor gives the appropriate degeneracies.

The Hamiltonian for the relative motion, which does depend on the statistics of the particles, can also be solved in the same way. In the boson gauge, we have

$$H_{\rm rel}\psi_{\rm rel} = [\frac{(\mathbf{p} - q\mathbf{a}_{\rm rel})^2}{m} + \frac{1}{4}m\omega^2 r^2]\psi_{\rm rel} = E_{\rm rel}\psi_{\rm rel} \tag{2.32}$$

with $\mathbf{a}_{\rm rel} = (0, \phi/2\pi r)$ and $\psi_{\rm rel}(r, \theta + \pi) = \psi_{\rm rel}(r, \theta)$. The wavefunction, once again, factorises in r and θ and is given by

$$\psi_{\rm rel}(r, \theta) = e^{il\theta}\mathcal{R}_{\rm rel}(r) \tag{2.33}$$

However, due to the boundary condition on $\psi_{\rm rel}$, l now has to be an even integer, analogous to the relative angular momentum quantum number for two free anyons . With the definition $k^2 = mE$, the radial equation is

$$[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}(l + \alpha/\pi)^2 - \frac{1}{4}m^2\omega^2 r^2 + k^2]\mathcal{R}_{\rm rel}(r) = 0, \tag{2.34}$$

where we have substituted $\alpha = q\phi/2$. As in the case of two free anyons, we see that the net

effect of statistics is to add a term to the centrifugal barrier. A series solution of the form

$$\mathcal{R}_{\text{rel}}(r) = e^{-m\omega r^2/4} \sum_{n=0}^{\infty} b_n r^{n+s} \tag{2.35}$$

can be found to Eq.(2.34), with $s = |l + \alpha/\pi|$ and

$$\frac{b_{n+2}}{b_n} = \frac{k^2 - m\omega(n + |l + \alpha/\pi| + 1)}{(n+2)^2 + 2|l + \alpha/\pi|(n+2)}, \tag{2.36}$$

so that n has to be an even integer. As before, the requirement that the series has to terminate leads to the energy levels given by

$$E_{\text{rel}} = \omega(n + |l + \alpha/\pi| + 1). \tag{2.37}$$

Notice that $\alpha = 0$ and $\alpha = \pi$ give the usual energy levels for bosons and fermions respectively. The first few energy levels and their degeneracies are explicitly given by

$$\begin{aligned} E_0 &= (1 + \alpha/\pi)\omega, && \text{deg} = 1, \\ E_1 &= (3 - \alpha/\pi)\omega, && \text{deg} = 1, \\ E_2 &= (3 + \alpha/\pi)\omega, && \text{deg} = 2, \\ E_3 &= (5 - \alpha/\pi)\omega, && \text{deg} = 2, \\ E_4 &= (5 + \alpha/\pi)\omega, && \text{deg} = 3. \end{aligned} \tag{2.38}$$

Thus, it is clear that the energy levels can be written as

$$\begin{aligned} & E_j = (2j + 1 + \alpha/\pi)\omega, && \text{deg} = j + 1 \\ \text{and} \quad & E_j = (2j + 1 - \alpha/\pi)\omega, && \text{deg} = j, \end{aligned} \tag{2.39}$$

where j is an integer. The energy levels can be plotted as a function of α as shown in Fig.(12).

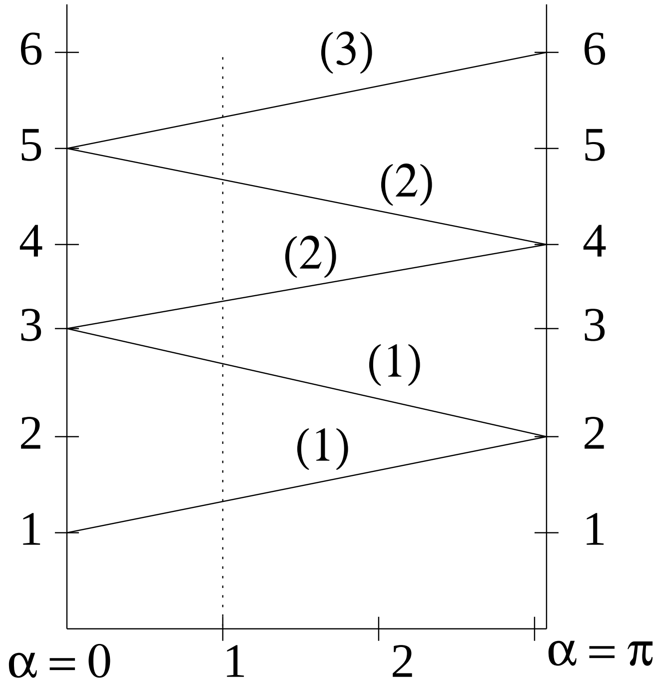


Fig. 12

Here $\alpha = 0$ corresponds to bosons and $\alpha = \pi$ corresponds to fermions. The degeneracies of the levels are mentioned within brackets. From the figure, it is clear that the energies of the 2 particle system are a monotonic function of the α and change continuously as α changes from bosonic to fermionic value. Moreover, except at $\alpha = 0, \pi/2, \pi$, the energy levels are not equally spaced and even at $\alpha = \pi/2$, the spacing between energy levels is half that for bosons and fermions. We also note that the energy levels do not cross each other as a function of α for this system. This property, however, does not necessarily hold even for other two anyon systems and, in general, systems with three or more anyons in any potential have level crossings.

Let us now combine the energy levels of the relative Hamiltonian with the energy levels of the CM system to obtain the full two particle energy levels - *i.e.*,

$$E_{\text{2particles}} = E_{\text{CM}} + E_{\text{rel}} = (2j + p + 2 \pm \alpha/\pi)\omega. \tag{2.40}$$

Now compare these energy levels with the energy levels

$$E_n = (n+1)\omega, \qquad n = \text{integer} \tag{2.41}$$

of a single particle in a two dimensional oscillator. For two particles, we would naively have expected the energy levels to be of the form

$$E_{n,m} = (n+1)\omega + (m+1)\omega \qquad n, m = \text{integers}. \tag{2.42}$$

This expectation is borne out only at $\alpha = 0$ and $\alpha = \pi$, where it is clear that Eq.(2.40) is of the form of Eq.(2.42), so that two particle energy levels can always be written in terms of sums of single particle energy levels. However, for arbitrary α, the two anyon energy levels bear no simple additive or combinatoric relation to the levels of a one anyon system. This is the root cause of the difficulty in handling many anyon systems.

A study of the ground state wavefunction also illuminates the same point. In the anyon gauge, the wavefunction is given by

$$\psi(R, \Theta, r, \theta) \propto e^{-m\omega(R^2+r^2/4)} r^{\alpha/\pi} e^{i\alpha\theta/\pi} = e^{-m\omega(R^2+r^2/4)} (\mathbf{r})^{\alpha/\pi} \tag{2.43}$$

where $\mathbf{r}$ in complex coordinates is $x + iy = r\cos\theta + ir\sin\theta$. This wavefunction can be written in the original two particle coordinate system as

$$\psi(\mathbf{r}_1, \mathbf{r}_2) \propto e^{-m\omega(r_1^2+r_2^2)/2} (\mathbf{r}_1 - \mathbf{r}_2)^{\alpha/\pi}, \tag{2.44}$$

which, incidentally, vanishes except for bosons ($\alpha = 0$), thereby proving that anyons obey the Pauli principle. It is easy to see that for $\alpha = 0$ and $\alpha = \pi$, this wavefunction can be factorised into a product of two single particle wavefunctions —

$$\alpha = 0 \qquad \psi(\mathbf{r}_1, \mathbf{r}_2) = e^{-m\omega r_1^2/2} e^{-m\omega r_2^2/2} = \chi_0(\mathbf{r}_1)\chi_0(\mathbf{r}_2) \tag{2.45}$$

and

$$\alpha = \pi \qquad \begin{aligned} \psi(\mathbf{r}_1, \mathbf{r}_2) &= \mathbf{r}_1 e^{-m\omega r_1^2/2} e^{-m\omega r_2^2/2} - \mathbf{r}_2 e^{-m\omega r_1^2/2} e^{-m\omega r_2^2/2} \\ &= \chi_1(\mathbf{r}_1)\chi_0(\mathbf{r}_2) - \chi_0(\mathbf{r}_1)\chi_1(\mathbf{r}_2) \end{aligned} \tag{2.46}$$

where χ_0 and χ_1 are the ground and first excited states of the one particle system. For arbitrary α, however, the two particle wavefunction bears no simple relation to the one particle wavefunctions. Thus, many anyon states are not products of single anyon states. This is why even a system of free anyons needs to be tackled as an interacting theory.

There are other simple two anyon problems that can be solved exactly, such as the problem of two anyons in a Coulomb potential [15]. However, no three anyon problem has been completely solved so far. The main hurdle in solving systems with three or more anyons is that the relative phases between any two anyons depends also on the positions of all the other anyons in the system. This makes the problem quite intractable. Only partial solutions have been obtained so far using various different methods. Exact solutions for a part of the spectrum of three or more free anyons or in a harmonic oscillator potential (or equivalently an external magnetic field) have been found [16]. A semiclassical approach to the computation of the energy levels [17] has also been attempted. Perturbative approaches [18] in the limit of small α, where α is the statistics parameter, have also been used to demonstrate level crossings and the piecewise continuity of the ground state [19]. The level crossing phenomenon has been confirmed by recent numerical computations which have been used to obtain the first twenty odd energy levels [20], which also showed that the analytic solutions found earlier formed a very small subset of the total number of solutions. Finally, there exist some exact results [21] regarding the symmetry of the spectrum as a function of the statistics parameter α. However, the computation of the full spectrum of energy levels of a three anyon system is beyond our capacity at the present time. Hence, we emphasize here that by and large, the study of three and more anyon quantum mechanics is still an open problem.

Problems

1. For a two-dimensional system of charged particles in a uniform magnetic field B (work in the gauge $\mathbf{A} = \frac{B}{2}(-y, x) \quad$), write down the Hamiltonian in terms of the complex coordinates z and $\bar{z}$ where $z = (x + iy)/\ell$ and $\bar{z} = (x - iy)/\ell$ and ℓ is the magnetic length defined to be $\ell = (\hbar c/eB)^{1/2} = (1/eB)^{1/2}$.

$a.$) Show that $[a, a^\dagger] = 1$ and $[b, b^\dagger] = 1$ where $a = \frac{1}{\sqrt{2}}(2\frac{\partial}{\partial \bar{z}} + \frac{z}{2})$ and $b = \frac{1}{\sqrt{2}}(2\frac{\partial}{\partial z} + \frac{\bar{z}}{2})$.

$b.$) Express the Hamiltonian in terms of the operators $a, a^\dagger, b$ and $b^\dagger$ and find the eigenvalues of H. What are the quantum numbers required to classify the states? How do you see the degeneracy of the states in terms of the quantum numbers?

2. Solve the problem of Mott scattering of anyons (scattering from a $1/r$ potential). If you get stuck, look up Ref [15].

3. Many Anyon Systems:Quantum Statistical Mechanics

In this section, we shall study the quantum mechanical system of many anyons using the virial expansion of the equation of state, which is valid in the high temperature, low density limit. However, before starting on the many anyon problem, we shall show how the second virial coefficient of a system of free fermions or bosons is obtained and review in some detail, the cluster expansion method, which leads to the expression for the virial coefficients in terms of the cluster integrals [22].

For bosons and fermions, quantum statistical mechanics begins with the calculation of the grand partition function. The canonical partition function is given by

$$Z(A,T) = \sum_{\text{states}} e^{-\beta E_N} \tag{3.1}$$

where E_N is the energy of the N-particle system. In terms of single particle energy levels $\epsilon_{\mathbf{p}}$, E_N can be written as

$$E_N = \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} n_{\mathbf{p}} \tag{3.2}$$

where $\mathbf{p}$ enumerates the energy levels and $n_{\mathbf{p}}$ is the occupation number of each level. The occupation number $n_{\mathbf{p}}$ is just (0,1) for fermions and (0,1,2,.....) for bosons, but it is constrained by the total number of particles given by

$$N = \sum_{\mathbf{p}} n_{\mathbf{p}}. \tag{3.3}$$

The canonical partition function cannot be easily evaluated because of this constraint. However, the grand canonical partition function defined by

$$Z_G(z,A,T) = \sum_{N=0}^{\infty} z^N Z(A,T) = \sum_{N=0}^{\infty} z^N \sum_{\substack{\{n_{\mathbf{p}}\} \\ \Sigma_{\mathbf{p}} n_{\mathbf{p}} = N}} e^{-\beta \Sigma_p \epsilon_{\mathbf{p}} n_{\mathbf{p}}} \tag{3.4}$$

where $\{n_{\mathbf{p}}\}$ represents the collection of values of $n_{\mathbf{p}}$ for the different states, invalidates the constraint, since it incorporates a sum over all N. Now, from Eq.(3.3), we see that

$z^N = \Pi_p z^{n_{\mathbf{p}}}$ and $e^{-\beta \Sigma_p \epsilon_{\mathbf{p}} n_{\mathbf{p}}} = \Pi_p (e^{-\beta\epsilon_{\mathbf{p}}})^{n_{\mathbf{p}}}$ yielding

$$Z_G(z, A, T) = \sum_{N=0}^{\infty} \sum_{\{n_{\mathbf{p}}\}} \prod_{\mathbf{p}} (z e^{-\beta\epsilon_{\mathbf{p}}})^{n_{\mathbf{p}}}. \tag{3.5}$$

The whole purpose of considering the grand partition function instead of the partition function becomes clear when we realise that the two sums over N and $\{n_{\mathbf{p}}\}$ reduce to independent sums over each $n_{\mathbf{p}}$ -*i.e.*,

$$Z_G(z, A, T) = \sum_{n_{\mathbf{p}}} \prod_{\mathbf{p}} (z e^{-\beta\epsilon_{\mathbf{p}}})^{n_{\mathbf{p}}} = \prod_{\mathbf{p}} \sum_{n} (z e^{-\beta\epsilon_{\mathbf{p}}})^{n}. \tag{3.6}$$

Thus, the grand partition function is given by

$$\begin{aligned} & Z_G(z, A, T) = \prod_{\mathbf{p}} (1 + z e^{-\beta\epsilon_{\mathbf{p}}}) \qquad \text{for} \quad \text{fermions}, \\ \text{and} \qquad & Z_G(z, A, T) = \prod_{\mathbf{p}} \frac{1}{(1 - z e^{-\beta\epsilon_{\mathbf{p}}})} \qquad \text{for} \quad \text{bosons}, \end{aligned} \tag{3.7}$$

respectively. This method is inapplicable to anyons, because, as we have seen in Sec.(2), the energy levels of an N-particle system cannot be computed in terms of the single particle energies. Hence, the exact calculation of the partition function for an anyon gas still remains an open problem.

However, there exist other approximation schemes under which quantum gases are studied, one of which is called the cluster expansion (CE) method. The classical CE involves a systematic expansion of the interparticle potential, which is valid in the high temperature, low density regime. The quantum CE is defined by analogy with the classical CE. Using this expansion, corrections to the ideal gas law by systems of interacting quantum particles can be computed. This method is available to the anyon gas as well, because anyons can always be thought of as interacting bosons or fermions.

Let us briefly review the cluster expansion method before applying it to the anyon gas. We consider a classical Boltzmann gas with interactions. The many particle Hamiltonian

is given by

$$H = \sum_i \frac{\mathbf{p}_i^2}{2m} + \sum_{i,j,...} V(\mathbf{r}_i, \mathbf{r}_j, ...) \tag{3.8}$$

and the canonical partition function for N particles, obtained by integrating over all of classical phase space, is

$$Z(A,T) = \frac{1}{(2\pi)^{2N} N!} \int d^2\mathbf{r}_1 .....d^2\mathbf{r}_N d^2\mathbf{p}_1 .....d^2\mathbf{p}_N e^{-\beta H}. \tag{3.9}$$

(Classically, the N particles are distinguishable and taken to be distinct. We divide by $N!$ to compensate for the overcounting.) The momentum integrations can be performed, since the integrand can be factorised into a product and the individual integrations are merely Gaussian -*i.e.*,

$$\int \frac{d^2\mathbf{p}}{(2\pi)^2} e^{-\frac{\beta \mathbf{p}^2}{2m}} = \frac{m}{2\pi\beta} \equiv \frac{1}{\lambda^2} \tag{3.10}$$

where λ is called the thermal wavelength. Thus, the canonical partition function, reduces to

$$Z(A,T) = \frac{1}{N!\lambda^{2N}} \int d^2\mathbf{r}_1 .....d^2\mathbf{r}_N e^{-\beta \Sigma_{i,j,..} V(\mathbf{r}_i, \mathbf{r}_j, ..)}. \tag{3.11}$$

Now, the potential can be separated into sums of n-body potentials - *i.e.*,

$$\sum_{i,j,..} V(\mathbf{r}_i, \mathbf{r}_j, ...) = \sum_{i,j} V(\mathbf{r}_i, \mathbf{r}_j) + \sum_{i,j,k} V(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) + ... \quad . \tag{3.12}$$

We shall see that the lowest order correction to the ideal gas law involves only two body interactions, but let us keep the formulation general at this stage. The next step is to expand the integrand of Eq.(3.11) in powers of $e^{-\beta\Sigma V} - 1$. We define

$$\begin{aligned} e^{-\beta V(\mathbf{r}_i, \mathbf{r}_j)} &\equiv (1 + f_{ij}) \\ e^{-\beta V(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k)} &\equiv (1 + f_{ijk}) \end{aligned} \tag{3.13}$$

and so on. Each of these $f_{ijk...}$ code for the n-particle interactions involved in the potential.

The partition function can be written in terms of the $f_{ijk..}$ as

$$
\begin{aligned}
Z(A,T) &= \frac{1}{N!\lambda^{2N}} \int d^2\mathbf{r}_1.....d^2\mathbf{r}_N \prod_{i<j}(1+f_{ij}) \prod_{i<j<k}(1+f_{ijk}) \prod_{i<j<k<l}(1+f_{ijkl}).... \\
&= \frac{1}{N!\lambda^{2N}} \int d^2\mathbf{r}_1.....d^2\mathbf{r}_N[1+(f_{12}+f_{23}+...)+(f_{12}f_{23}+f_{12}f_{14}+...)+...] \\
&\qquad + (f_{12}f_{23}f_{13}+f_{12}f_{24}f_{14}+...)+(f_{123}+f_{124}+...)+....
\end{aligned}
\tag{3.14}
$$

Now let us study, in more detail, each l-cluster involving interactions between l particles. A cluster integral b_l is defined as

$$
b_l = \frac{1}{l!\lambda^{2l-2}A} \int d^2\mathbf{r}_1.....d^2\mathbf{r}_N(\text{contribution} \quad \text{from} \quad l-\text{clusters}) \tag{3.15}
$$

Explicitly,

$$
b_1 = \frac{1}{A}\int d^2\mathbf{r}_1.1 = 1, \tag{3.16}
$$

$$
\begin{aligned}
b_2 &= \frac{1}{2\lambda^2 A}\int d^2\mathbf{r}_1 d^2\mathbf{r}_2 f_{12} \\
&= \frac{1}{2\lambda^2 A}\int d^2\mathbf{r}_1 d^2\mathbf{r}_3 f_{13} \\
&= \frac{1}{2\lambda^2 A}\int d^2\mathbf{r}_i d^2\mathbf{r}_j f_{ij},
\end{aligned}
\tag{3.17}
$$

where the last equality is valid for any two arbitrary particles labelled i and j, and

$$
b_3 = \frac{1}{6\lambda^4 A}\int d^2\mathbf{r}_1 d^2\mathbf{r}_2 d^2\mathbf{r}_3(f_{12}f_{13}+f_{13}f_{23}+f_{12}f_{23}+f_{12}f_{13}f_{23}+f_{123}) \tag{3.18}
$$

with similar contributions from any other choice of three particles. Notice that the contributions obtained by changing the particle indices are all equal. Also note that the computation of b_2 involves only two-body interactions even if the Hamiltonian includes three or more body interactions.

It is now clear that the partition function may be written as a product of cluster integrals -*i.e.*,

$$
Z(A,T) \sim \frac{1}{N!\lambda^{2N}} \sum_{\{m_l\}} \prod_l (l!\lambda^{2l-2}A)^{m_l} b_l^{m_l} \tag{3.19}
$$

with $\Sigma l m_l = N$. Here m_l denotes the number of l-clusters and $\{m_l\} = (m_1, m_2, ...)$ denotes

the collection of m_l. However, we still need the combinatoric factor, which specifies the number of times a given cluster integral appears in the product. Let us compute this combinatoric factor. Firstly, since there are N particles in the system, naively we expect $N!$ clusters. Hence, the $R.H.S$ of Eq.(3.19) is multiplied by $N!$. However, particles in the same cluster are indistinguishable; so we have to divide by $(l!)^{m_l}$. Moreover, any two l-clusters are indistinguishable. Hence, for m_l l-clusters, we have to divide the $R.H.S$ by $m_l!$. Putting all this together, the partition function is given by

$$\begin{aligned} Z(A,T) &= \frac{1}{N!\lambda^{2N}} \sum_{\{m_l\}} \prod_l (l!\lambda^{2l-2}A)^{m_l} \frac{N!}{(l!)^{m_l} m_l!} b_l^{m_l} \\ &= \sum_{\{m_l\}} \prod_l \frac{1}{m_l!} (\frac{Ab_l}{\lambda^2})^{m_l} \end{aligned} \tag{3.20}$$

with $\Sigma l m_l = N$. (The simplest way, to convince yourself that this expression for the partition function in terms of the cluster integrals and all the combinatoric factors is correct, is to work out the partition function for a few particle systems explicitly.) As before, the constraint on the number of particles makes this partition function difficult to evaluate. Hence, just as was done for fermions and bosons, we move on to the grand partition function where the total number of particles N is also summed over. The expression for the grand partition function for this system is given by

$$\begin{aligned} Z_G(z,A,T) &= \sum_{N=0}^{\infty} z^N Z(A,T) \\ &= \prod_l \sum_{m_l=0}^{\infty} \frac{1}{m_l!} (\frac{Ab_l z^l}{\lambda^2})^{m_l} \\ &= e^{\Sigma_l (\frac{Ab_l z^l}{\lambda^2})}. \end{aligned} \tag{3.21}$$

Notice that here again, the unrestricted sum over N and the sum over $\{m_l\}$ has been translated to an unrestricted sum over each of the m_l. The equation of state obtained from this partition function is given by

$$\frac{PA}{KT} = \ln Z_G = \frac{A}{\lambda^2} \sum_l b_l z^l. \tag{3.22}$$

Furthermore, the average number of particles $< N >= N$ is given by

$$N = z\frac{\partial}{\partial z}\ln Z_G = \frac{A}{\lambda^2}\sum_l l b_l z^l. \tag{3.23}$$

Combining Eq.(3.22) and Eq.(3.23), we get

$$\frac{PA}{NKT} = \frac{\Sigma_l b_l z^l}{\Sigma_l l b_l z^l} = \frac{b_1 z + b_2 z^2 + ...}{b_1 z + 2b_2 z^2 + ...}. \tag{3.24}$$

Since $b_1 = 1$, to first order in z, we find that

$$\frac{PA}{NKT} = 1 - b_2 z + O(z^2). \tag{3.25}$$

Now, the virial coefficients of the system are defined by

$$\frac{PA}{NKT} = \frac{P}{\rho KT} = \sum_{l=1}^{\infty} a_l (\rho\lambda^2)^{l-1} \tag{3.26}$$

where ρ is the density of particles. Thus, we have a power series expansion of the equation of state in terms of the inverse temperature λ and the density ρ. This is clearly a useful concept at high temperatures and low densities, where only the first few terms in the series are likely to be relevant. Substituting for $(\rho\lambda^2)$ from Eq.(3.23), we see that

$$\begin{aligned} \frac{P}{\rho KT} &= \sum_{l=1}^{\infty} a_l (\Sigma_{l'=1}^{\infty} b_{l'} l' z^{l'})^{l-1} \\ &= a_1 + a_2(z + 2b_2 z^2 + ...) + ... \end{aligned} \tag{3.27}$$

to lowest order in z. Comparing this equation with Eq.(3.25), we see that $a_1 = 1$ and $a_2 = -b_2$. So to find the second virial coefficient, we only need to compute the two-cluster integral, which, in turn, depends only on two-body interactions.

The quantum cluster expansion is defined by analogy with the classical cluster expansion. The quantum partition function is given by

$$\begin{aligned} Z &= \sum_{\alpha} \int d^2\mathbf{r}_1.....d^2\mathbf{r}_N \psi_\alpha^*(\mathbf{r}_1, ...\mathbf{r}_N) e^{-\beta H} \psi_\alpha(\mathbf{r}_1, ...\mathbf{r}_N) \\ &= \frac{1}{N!\lambda^{2N}} \int d^2\mathbf{r}_1.....d^2\mathbf{r}_N W_N(\mathbf{r}_1, ...\mathbf{r}_N), \end{aligned} \tag{3.28}$$

where $\{\psi_\alpha\}$ is a complete set of orthonormal wavefunctions for the system labelled by the quantum number α and the last equality defines $W_N(\mathbf{r}_1, ...\mathbf{r}_N)$. By comparing this equation

with the definition of the classical partition function in Eq.(3.11), we see that

$$W_N^{\rm cl}(\mathbf{r}_1,...\mathbf{r}_N) = e^{-\beta\Sigma_{i,j}V(\mathbf{r}_i,\mathbf{r}_j)} = \prod_{i<j}(1+f_{ij}) \tag{3.29}$$

(Since we shall only compute the second virial coefficient and we have seen that the second virial coefficient depends only on two body interactions, we have specialised to the case of two-body interactions alone.) Now, let us define new quantities $U_l(\mathbf{r}_1,...\mathbf{r}_l)$ through

$$\begin{aligned} W_1(\mathbf{r}_1) &= U_1(\mathbf{r}_1), \\ W_2(\mathbf{r}_1,\mathbf{r}_2) &= U_1(\mathbf{r}_1)U_1(\mathbf{r}_2) + U_2(\mathbf{r}_1,\mathbf{r}_2), \\ W_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) &= U_1(\mathbf{r}_1)U_1(\mathbf{r}_2)U_1(\mathbf{r}_3) + U_1(\mathbf{r}_1)U_2(\mathbf{r}_2,\mathbf{r}_3) + U_1(\mathbf{r}_2)U_2(\mathbf{r}_1,\mathbf{r}_3) \\ &\quad + U_1(\mathbf{r}_3)U_2(\mathbf{r}_1,\mathbf{r}_2) + U_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3), \end{aligned} \tag{3.30}$$

and so on. From Eq.(3.29), we see that the classical limits of the U_l functions can be identified as follows —

$$\begin{aligned} W_1^{\rm cl}(\mathbf{r}_1) &= 1 \Rightarrow U_1^{\rm cl}(\mathbf{r}_1) = 1, \\ W_2^{\rm cl}(\mathbf{r}_1,\mathbf{r}_2) &= 1.1 + f_{12} \Rightarrow U_2(\mathbf{r}_1,\mathbf{r}_2) = f_{12}, \\ W_3^{\rm cl}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) &= 1.1.1 + 1.(f_{12}+f_{13}+f_{23}) + (f_{12}f_{13}+f_{12}f_{23}+f_{13}f_{23}) + f_{12}f_{23}f_{13} \\ &\Rightarrow U_3^{\rm cl}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) = (f_{12}f_{13}+f_{12}f_{23}+f_{13}f_{23}) + f_{12}f_{13}f_{23}, \end{aligned} \tag{3.31}$$

and so on - *i.e.*, the $U_l(\mathbf{r}_1,...\mathbf{r}_l)$ are the quantum analogs of the l-clusters in the classical case. Hence, for quantum statistical mechanics, the classical cluster integrals can be replaced by the quantum cluster integrals given by

$$b_l = \frac{1}{l!\lambda^{2l-2}A}\int d^2\mathbf{r}_1.....d^2\mathbf{r}_N U_l(\mathbf{r}_1,...\mathbf{r}_l). \tag{3.32}$$

To calculate the second virial coefficient for any system, we need b_2, since we have already found that $a_2 = -b_2$, and to find b_2, we need to compute $W_2(\mathbf{r}_1,\mathbf{r}_2)$ which is a property of the two-body system. Let the Hamiltonian for the two-body system be

$$H = -\frac{1}{2m}(\nabla_1^2+\nabla_2^2) + v(|\mathbf{r}_1-\mathbf{r}_2|) \tag{3.33}$$

with eigenvalues E_α and eigenfunctions $\psi_\alpha(\mathbf{r}_1,\mathbf{r}_2)$. We transform to the CM ($\mathbf{R}$) and

relative ($\mathbf{r}$) coordinates to solve the problem. In terms of these coordinates,

$$\psi_\alpha(\mathbf{r}_1, \mathbf{r}_2) \longrightarrow \psi_\alpha(\mathbf{R}, \mathbf{r}) = \frac{e^{i\mathbf{P}\cdot\mathbf{R}}}{\sqrt{A}}\psi_n(\mathbf{r}) \tag{3.34}$$

and $E_\alpha = \mathbf{P}^2/4m + \epsilon_n$. Here, the quantum number α has been split into $(\mathbf{P}, n)$ where $\mathbf{P}$ refers to the continuum quantum numbers of the CM system and n is the quantum number labelling the energy levels of the relative Hamiltonian. ϵ_n is found by solving the eigenvalue equation of the relative Hamiltonian given by

$$[-\frac{1}{m}\nabla^2 + v(\mathbf{r})]\psi_n(\mathbf{r}) = \epsilon_n(\mathbf{r})\psi_n(\mathbf{r}). \tag{3.35}$$

The definition of $W_2(\mathbf{r}_1, \mathbf{r}_2)$ in Eq.(3.28) leads to its identification in this system as

$$\begin{aligned} W_2(\mathbf{r}_1, \mathbf{r}_2) &= 2\lambda^4 \sum_\alpha \psi^*_\alpha(\mathbf{r}_1, \mathbf{r}_2) e^{-\beta H}\psi_\alpha(\mathbf{r}_1, \mathbf{r}_2) \\ &= \frac{2A\lambda^4}{(2\pi)^2}\int d^2\mathbf{P}\sum_n \frac{e^{-\beta\mathbf{P}^2/4m - \beta\epsilon_n}}{A}|\psi_n(\mathbf{r})|^2 \\ &= 4\lambda^2 \sum_n |\psi_n(\mathbf{r})|^2 e^{-\beta\epsilon_n} \end{aligned} \tag{3.36}$$

which, in turn, identifies

$$U_2(\mathbf{r}_1, \mathbf{r}_2) = 4\lambda^2 \sum_n |\psi_n(\mathbf{r})|^2 e^{-\beta\epsilon_n} - 1. \tag{3.37}$$

Hence, the definition of the quantum cluster integral in Eq.(3.32) leads to

$$b_2 = -a_2 = \frac{1}{2A\lambda^2}\int d^2\mathbf{R}d^2\mathbf{r}[4\lambda^2 \sum_n |\psi_n(\mathbf{r})|^2 e^{-\beta\epsilon_n} - 1]. \tag{3.38}$$

The two individual terms in the integrand above give rise to area divergences when integrated over $\mathbf{r}$ and $\mathbf{R}$. It is only their difference that is finite in the thermodynamic limit. (Even the difference is not always finite. The usual condition for finiteness of the difference is that the interaction between particles should be short-ranged.) Often, it is more convenient to compute the second cluster integral as a difference between b_2 for the system

under study and b_2^0 for some reference system, where b_2^0 is known exactly - *e.g.*, b_2^0 could be computed when $v(\mathbf{r}) = 0$. Then , for normalised relative wavefunctions ,-*i.e.*, when $\int d^2\mathbf{r}|\psi_n(\mathbf{r})|^2 = 1$, we get

$$b_2 - b_2^0 = 2\sum_n (e^{-\beta\epsilon_n} - e^{-\beta\epsilon_n^0}). \tag{3.39}$$

For the ideal Fermi and Bose gases, b_2^0 can be directly identified from the equation of state. For the Fermi gas, from Eq.(3.7), we see that the equation of state is given by

$$\frac{PA}{KT} = \ln Z_G = \frac{A}{(2\pi)^2}\int d^2\mathbf{p}\log(1 + ze^{-\beta\mathbf{p}^2/2m}), \tag{3.40}$$

since for free fermions $\epsilon_{\mathbf{p}} = \mathbf{p}^2/2m$. The $R.H.S$ can now be expanded in a power series in z from which the b_l may be identified - *i.e.*,

$$\begin{aligned}\frac{PA}{KT} &= \frac{A}{\lambda^2}(z - \frac{z^2}{2^2} + \frac{z^3}{3^2} - ...)\\ &= \frac{A}{\lambda^2}\sum_l b_l^0 z^l.\end{aligned} \tag{3.41}$$

Hence, b_2^0(fermions) is identified as $b_{2f}^0 = -1/4$. Similarly, for the Bose gas,

$$\begin{aligned}\frac{PA}{KT} &= -\frac{A}{(2\pi)^2}\int d^2\mathbf{p}\log(1 - ze^{-\beta\mathbf{p}^2/2m})\\ &= \frac{A}{\lambda^2}(z + \frac{z^2}{2^2} + \frac{z^3}{3^2} + ...)\end{aligned} \tag{3.42}$$

so that b_2^0(bosons) is identified as $b_{2b}^0 = 1/4$.

We had earlier noted that the free anyon gas is already an interacting system, so a direct evaluation of the partition function is not possible. However, the second virial coefficient can be computed as long as we know the energy levels of the two anyon system. In Sec.(2), we explicitly computed the energy levels of two anyons in a harmonic oscillator potential. Let us use those results to find the second virial coefficient of an anyon gas in the same harmonic oscillator potential [11]. Ultimately, we shall take the limit where the oscillator

potential vanishes to obtain the virial coefficient of the free anyon gas. We shall choose our reference system to be that of free bosons ($\alpha = 0$). From Eq.(2.39), we find that

$$
\begin{aligned}
b_2 - b_{2b}^0 &= 2\sum_{j=0}^{\infty}[(j+1)e^{-\beta(2j+1+\alpha/\pi)\omega} + je^{-\beta(2j+1-\alpha/\pi)\omega} - (j+1)e^{-\beta(2j+1)\omega} - je^{-\beta(2j+1)\omega}] \\
&= 4[\cosh(\alpha/\pi - 1)\beta\omega - \cosh\beta\omega]\sum_{j=0}^{\infty} je^{-2j\beta\omega} \\
&= \frac{\cosh(\alpha/\pi - 1)\beta\omega - \cosh\beta\omega}{\sinh^2\beta\omega}.
\end{aligned}
\tag{3.43}
$$

To find the virial coefficient of a system of free anyons, we take the limit $\omega \to 0$, which yields

$$
b_2 - b_{2b}^0 = \frac{1}{2}[(\frac{\alpha}{\pi})^2 - \frac{2\alpha}{\pi}], \tag{3.44}
$$

so that substituting $b_{2b}^0 = 1/4$, we get

$$
b_2 = \frac{1}{4}[2(\frac{\alpha}{\pi})^2 - \frac{4\alpha}{\pi} + 1]. \tag{3.45}
$$

Notice that this equation is only valid for $0 \leq \alpha < 2\pi$, since α being a periodic variable specifying the statistics is periodic in 2π. Hence, the virial coefficient is non-analytic in α and has a cusp whenever $\alpha = 2\pi j$ for j an integer. By defining δ such that $\alpha = 2\pi j + \delta$, we see that

$$
b_2 = \frac{1}{4}[2(\frac{\delta}{\pi})^2 - \frac{4\delta}{\pi} + 1] \tag{3.46}
$$

is valid for any α. The virial coefficient can be plotted [10] as a function of α as shown in Fig.(13).

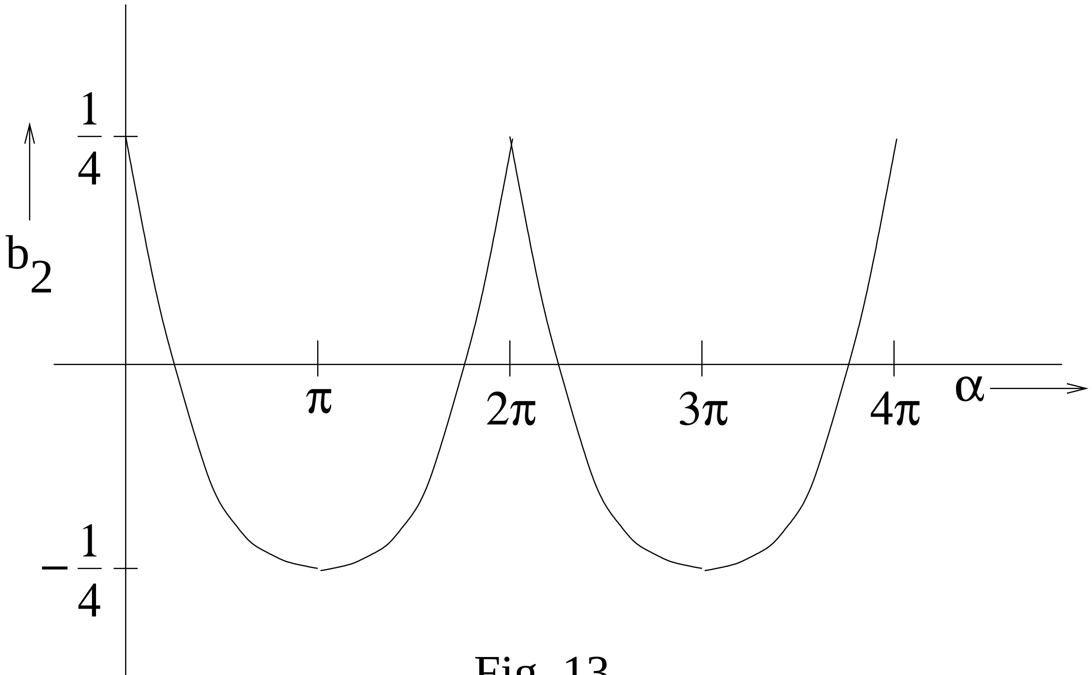


Fig. 13

From the figure, it is clear that the values of the virial coefficient of an anyon gas interpolates between the values of that for fermions and bosons.

Our result for the virial coefficient of an anyon gas is not obvious, though it is satisfying - just as spin and statistics of anyons are intermediate between spin and statistics of fermions and bosons, so are the virial coefficients. In fact, it is not even clear why the cluster expansion method works, because, when anyons are considered as interacting fermions or bosons, even the two-body interaction between particles is actually long range. However, the result that we have obtained here for the anyon gas is consistent, because the same answer has also been obtained using other regularisation schemes. Here, we used the harmonic oscillator potential as a regulator to discretise the energy levels and then took the limit where the oscillator potential vanishes to obtain the virial coefficient, whereas the original calculation of the virial coefficient involved a box normalisation [10].

The next logical step in this programme would be to compute the third virial coefficient. But, this would require the knowledge of three-body interactions. Since, as was mentioned in Sec.(2), the three anyon problem has not been completely solved in any potential so far,

the computation of the third virial coefficient too is an unsolved problem. However, as in the case of three anyon quantum mechanics, partial results have been obtained [23].

Problems

1. In class, we studied the virial coefficient of a gas of anyons using a harmonic oscillator potential as a regulator. Show that the same answer is obtained using box regularisation. (This was how it was originally done in Ref.[10].)

4. Many Anyon Systems : Mean Field Approach

The basic idea of the mean field approach is to replace the effect of many particles by an 'average' or 'mean' field and to accomodate deviations from the mean field as residual short range interactions. In the context of anyons, the mean field approach involves replacing the flux-tubes carried by the charges by a uniform magnetic field with the same flux density [5] [6]. It is clear that this approximation is valid when the density of flux-tubes (equivalently particles) is high and fluctuations are small, -*i.e.*, it is a high density, low temperature expansion.

The many anyon Hamiltonian can be obtained by generalising the two anyon Hamiltonian in Eq.(2.1) to N particles as

$$H = \sum_{i=1}^{N} \frac{(\mathbf{p}_i - q\mathbf{a}_i)^2}{2m} \tag{4.1}$$

with

$$\mathbf{a}_i = \frac{\phi}{2\pi} \sum_{i \neq j} \frac{\hat{z} \times (\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2}. \tag{4.2}$$

Thus the charge in each anyon sees the vector potential due to the flux-tubes in all the other anyons. We can also compute the magnetic field at the position of the i^{th} charge. However, a naive computation leads to

$$b_i = \nabla \times \mathbf{a}_i = 0, \tag{4.3}$$

which is not surprising since $\mathbf{a}_i$ can also be written as a gradient - *i.e.*,

$$\mathbf{a}_i = \frac{\phi}{2\pi} \sum_{i \neq j} \nabla_i \theta_{ij} \tag{4.4}$$

where θ_{ij} is the angle made by the vector $(\mathbf{r}_i - \mathbf{r}_j)$ with an arbitrary axis. But using a

regularisation scheme [24] with

$$\mathbf{a}_i = \lim_{\epsilon \to 0} \frac{\phi}{2\pi} \sum_{i \neq j} \frac{\hat{z} \times (\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2 + \epsilon^2}, \tag{4.5}$$

we can show that

$$b_i = \nabla \times \mathbf{a}_i = \phi \sum_{i \neq j} \delta(\mathbf{r}_i - \mathbf{r}_j). \tag{4.6}$$

Not surprisingly, there is no magnetic field at the position of the i^{th} charge, due to any other particle, unless the two particles coincide. However, in the mean field approach, the flux-tubes are replaced by a constant magnetic field b with the same flux density. Let us assume that the density of anyons per unit area is given by $\bar{\rho}$. Then the flux density of the anyons is given by

$$\int b_i d^2\mathbf{r} = \phi(\bar{\rho} - 1) \simeq \phi\bar{\rho} \tag{4.7}$$

when the integral is over a unit area and when $\bar{\rho}$ is sufficiently large. Hence, the appropriate uniform magnetic field to be used in the mean field approach is just

$$b = \phi\bar{\rho} \tag{4.8}$$

so that we have a system of charges moving in a constant magnetic field as illustrated in Fig.(14).

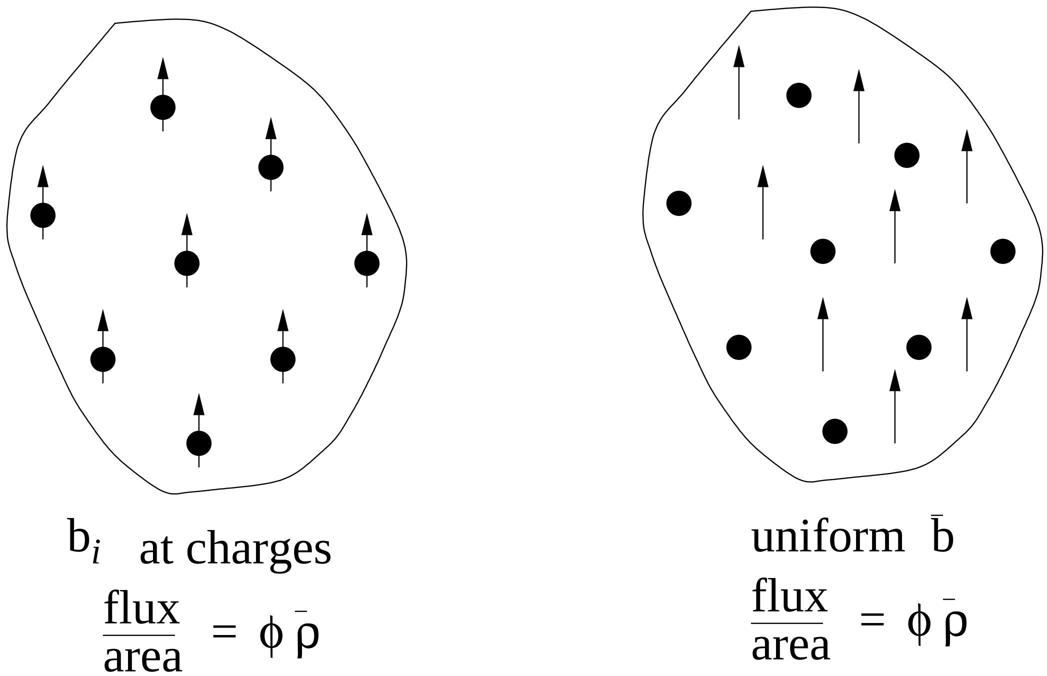


Fig. 14

Let us now study the quantum mechanical motion of a single particle in a constant magnetic field. The Hamiltonian is given by

$$H = \frac{(\mathbf{p} - q\mathbf{a})^2}{2m} \tag{4.9}$$

with the magnetic field $B = \nabla \times \mathbf{A} =$ constant. $\mathbf{A}$ can be chosen in many different ways (corresponding to different gauges), all of which lead to the same B. We shall work in the Landau gauge given by

$$A_x = -By, \quad A_y = 0. \tag{4.10}$$

(Another illustrative gauge (given as a problem in Chapter 2) is the symmetric gauge $A_x = -By/2, A_y = Bx/2$, which is useful in the study of the Fractional Quantum Hall phenomenon.) Therefore, the appropriate Schrodinger equation that governs the motion of the particle is given by

$$[\frac{(p_x + qBy)^2}{2m} + \frac{p_y^2}{2m}]\psi_{p_x,p_y}(x,y) = E_{p_x,p_y}\psi_{p_x,p_y}(x,y). \tag{4.11}$$

Here the p_x and p_y labels on the wavefunction and energy eigenvalues are the momentum

quantum numbers. Since there is no explicit x-dependence in the Hamiltonian, the motion in the x-direction is free and the wavefunction can be chosen to be of the form

$$\psi_{p_x,p_y}(x,y) = e^{ip_x x}\chi(y) \tag{4.12}$$

where $\chi(y)$ satisfies Eq.(4.11) where, however, p_x is now interpreted as the eigenvalue of the x-momentum operator. By defining $qB/m = \omega$ (ω is the cyclotron frequency of a charged particle moving in the magnetic field B), and the magnetic length $l = \sqrt{1/qB} = \sqrt{1/m\omega}$, Eq.(4.11) may be rewritten as

$$\frac{\omega}{2}[p_y^2 l^2 + (\frac{y}{l} + p_x l)^2]\chi(y) = E_{p_x,p_y}\chi(y). \tag{4.13}$$

This is just the the Schrodinger equation of a shifted harmonic oscillator in the (y, p_y) co-ordinates. Hence, the energy eigenvalues are discrete and given by

$$E_{p_x,n} = (n + 1/2)\omega, \tag{4.14}$$

where we have replaced the continuum label p_y in the subscript of E by the discrete numbers n. These discrete energy eigenvalues labelled by the integers n are called Landau levels. The corresponding eigenfunctions are given by

$$\chi(y) = \frac{1}{\pi^{1/4}}\frac{1}{\sqrt{2^n n! l}}e^{-\frac{(y/l+p_x l)^2}{2}}H_n(y/l + p_x l) \tag{4.15}$$

where the H_n are Hermite polynomials. Notice that the energy eigenvalues are independent of p_x, which only effects a shift in the origin of the oscillator. Since for motion in a plane, p_x is unrestricted, the Landau levels are infinitely degenerate. However, the degree of degeneracy becomes finite when the motion in the plane is restricted to a finite box with area $A = L_x L_y$. This degeneracy is easily computed. For motion in a one dimensional box of length L_x, p_x is quantised as

$$p_x = 2\pi n_x/L_x. \tag{4.16}$$

Furthermore, the allowed values of p_x are restricted by the condition that the centre of the

oscillator has to lie between 0 and L_y. Hence,

$$\frac{2\pi n_x}{L_x} l^2 < L_y \tag{4.17}$$

which, in turn, implies that the number of allowed values of n_x, or equivalently, the degeneracy of the Landau levels per unit area is given by

$$\frac{n_x}{L_x L_y} = \frac{1}{2\pi l^2} = \frac{qB}{2\pi}. \tag{4.18}$$

Let us now return to the many anyon problem which had earlier been reduced, in a mean field approach, to the problem of fermions or bosons moving in a uniform magnetic field. We shall choose our anyons to be fermions [6], rather than bosons, with attached flux-tubes, mainly because the problem of bosons in a magnetic field is itself unsolved. Hence, the analysis of fermions in a magnetic field (leading to anyon superconductivity) is easier. The other reason is that many properties of anyons appear to show a cusp in their behaviour in the bosonic limit. This was seen in Sec.(2), in the problem of two anyons in a harmonic oscillator potential, as well as in Sec.(3), where the second virial coefficient of the anyon gas was computed. Hence, the anyon gas may not have a smooth limit as the statistics parameter goes to zero if we start by perturbing from the bosonic end. However, the results that we shall obtain here by considering anyons to be fermion with attached flux-tubes have also been argued by starting with anyons as bosons [25] albeit with some approximations and hand-waving. Ultimately, whether we start with anyons as bosons or fermions is a matter of choice, and for each specific property of the anyon gas, one or the other approach may be easier.

In the mean field approach, we need to solve the problem of fermions moving in a uniform magnetic field $b = \phi\bar{\rho} = 2\alpha\bar{\rho}/q$. (Remember that in Sec.(2), we had shown that the statistics factor α is given by $q\phi/2$.) The degeneracy of the Landau levels in this field is given by

$$\text{deg} = \frac{qb}{2\pi} = \frac{\alpha\bar{\rho}}{\pi}. \tag{4.19}$$

Now, let us choose the statistics parameter α to be of the form $\alpha = \pi/n$, where n is any

integer, so that the degeneracy is simply

$$\deg = \frac{\bar{\rho}}{n}. \tag{4.20}$$

Since $\bar{\rho}$ is the density of particles and each level can contain $\bar{\rho}/n$ particles, clearly n Landau levels are completely filled. The next available single particle state is in the next Landau level which is an energetic distance $\omega = qb/m$ away. In the many-body or condensed matter parlance, this is called having a gap to single particle excitations. But if n is not an integer, then the last Landau level will only be partially filled and there will be no gap to single particle excitations. Hence, the parameter fractions $\alpha = \pi/n$ appear to be special (*e.g.*, like the magic numbers in the shell models of atomic and nuclear physics) and hence the states formed at these fractions should be particularly stable.

To prove that the states formed at these special fractions are superconducting, we have to study the effect of adding a real magnetic field B to the fictitious magnetic field b and check whether a Meissner effect exists. The argument differs slightly depending on the relative signs of B and b, and we shall consider both the cases separately.

When the real magnetic field is aligned parallel to the fictitious magnetic field, they add and increase the degeneracy of the Landau level - *i.e.*,

$$\deg = \frac{q(b+B)}{2\pi}. \tag{4.21}$$

But the number of particles per unit area $\bar{\rho} = nqb/2\pi$ remains unchanged. Hence, now the highest Landau level is only partially filled. Let us denote its filling fraction by $(1-x)$. From the conservation of density of particles, we have

$$(n-1)\frac{q(b+B)}{2\pi} + \frac{q(b+B)}{2\pi}(1-x) = \bar{\rho} = \frac{nqb}{2\pi} \tag{4.22}$$

from which we see that

$$(b+B)x = Bn. \tag{4.23}$$

Also from the energy eigenvalues in Eq.(4.14), we see that the total energy of $\bar{\rho}$ particles is

given by

$$E = \frac{q(b+B)}{2\pi} \sum_{j=0}^{n-2} (j+\frac{1}{2})\omega + \frac{q(b+B)}{2\pi}(n-\frac{1}{2})(1-x)\omega, \tag{4.24}$$

which can be simplified to give

$$E = \frac{q(b+B)}{2\pi} \frac{q(b+B)}{m} [\frac{n^2}{2} - (n-\frac{1}{2})x]. \tag{4.25}$$

Substituting for x from Eq.(4.23), we get

$$E = \frac{q^2 n^2}{4\pi m}[b^2 + \frac{bB}{n} - B^2(1-\frac{1}{n})] \tag{4.26}$$

For small external magnetic fields B, the energy relative to the ground state with no magnetic field is clearly positive and grows linearly with B. Thus, the anyon gas is a perfect diamagnet and tends to expel any external flux.

When the external magnetic field B is in the opposite direction to statistical magnetic field b, the degeneracy of Landau levels decreases - *i.e.*,

$$\mathrm{deg} = \frac{q(b-B)}{2\pi}. \tag{4.27}$$

So some of the $\bar{\rho}$ particles have to occupy the $(n+1)^{th}$ Landau level. Let us denote the filling fraction of the highest level by x. Then from conservation of particles, we have

$$\frac{nq(b-B)}{2\pi} + \frac{q(b-B)x}{2\pi} = \frac{nqb}{2\pi} \tag{4.28}$$

leading to

$$(b-B)x = Bn. \tag{4.29}$$

The total energy of the system is given by

$$E = \frac{q^2 n^2}{4\pi m}[b^2 + \frac{bB}{n} - B^2(1+\frac{1}{n})] \tag{4.30}$$

Notice that the linear term remains the same for B parallel and anti-parallel to b. Hence, once again for small B, the energy of this state relative to the state with $B = 0$ is positive and grows linearly with B, emphasizing the need to repel any external magnetic field.

We have just demonstrated the Meissner effect by showing that the anyon gas finds it energetically favourable to exclude any external magnetic field. Thus, the anyon gas is a superconductor. The effect of adding an external magnetic field to the anyon gas can be depicted schematically as shown in Fig.(15). (In the figure, crosses denote particles and circles denote holes.)

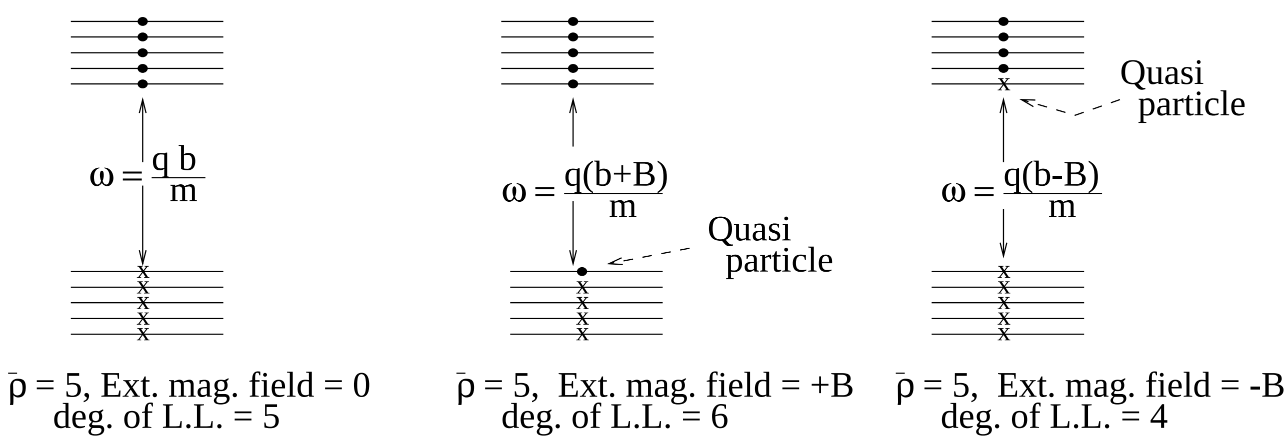


Fig. 15

Compare this with the schematic depiction (see Fig.(16)) of the creation of a quasiparticle-quasihole pair in the same system which costs an energy qb/m.

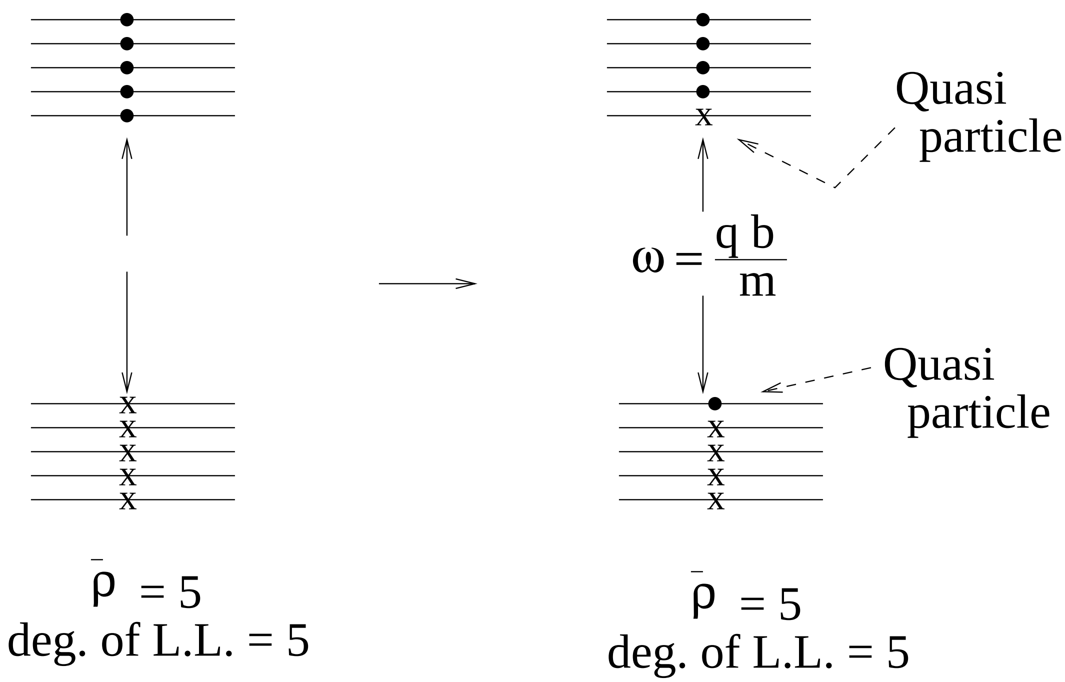


Fig. 16

It is clear that the production of quasiparticle and quasiholes are closely related to the presence of a real magnetic field, because if the quasiparticle and quasihole were spatially separated, then the quasiparticle excitation is analogous to the situation depicted in Fig.(15c) with a real magnetic field $-B$ and the quasihole excitation is analogous to the situation in Fig.(15b) with the field $+B$. (For $\bar{\rho}$ sufficiently large, the difference between $\bar{\rho}$ and $\bar{\rho} \pm 1$ is negligible.) With this connection, the Meissner effect can also be argued in the following way. Since for anyons, the magnetic field and hence degeneracy of Landau levels is tied to the density, to accomodate any external magnetic field which changes the degeneracy, particles (or holes) have to be excited across the gap. This costs energy and hence, penetration by magnetic fields is unfavourable. Conversely, we also see that if particles do not fill Landau levels, there must be a real magnetic field to account for the mismatch between density of particles and degeneracy of Landau levels. Hence, this argument also implies that every quasiparticle excitation in the system is accompanied by a real magnetic field, so that in anyon superconductors, charged quasiparticle excitations and vortex excitations are indistinguishable, in contrast to usual superconductors where

there are two types of excitations.

To actually prove superconductivity in the anyon gas, we need to show that the collective excitation in the system is massless. This can be done by including fluctuations (residual interactions) about the mean field state [5] [6] in a random phase approximation. Relativistic field theory models have also been used to obtain the massless mode [26]. However, even without going into the details of the calculation, there exists a simple heuristic argument to indicate the presence of the massless collective mode. Consider a very long wavelength density fluctuation (a collective excitation). Then $\bar{\rho}$, though varying, is approximately constant over macroscopic lengths. Within each such macroscopic area, $b = \alpha\bar{\rho}/\pi = \bar{\rho}/n$ is constant. Hence, locally the system always has n filled Landau levels. Therefore, such an excitation neither requires any particle to be excited into a higher Landau level nor does it require any energy. Hence, such a wave is massless - *i.e.*, we have proved the existence of a massless collective mode.

Much further work has been accomplished in this field. Important questions like the persistence of superconducting currents at finite temperatures have been addressed [27], though the results are not yet conclusive. The connection between superconducting anyon states and FQHE states have been explored [28] and for interacting anyons, new superconducting states have been found [29] at statistics parameter fractions $\alpha = \pi/\nu$, where ν is a fraction at which FQHE occurs. Also, the all-important question of experimental predictions and tests of anyon superconductors have been studied [30]. On the experimental front, the one robust experimental prediction of all anyon models of superconductivity has been violation of the discrete symmetries parity P and time reversal T. But recent experimental results [31] (though somewhat controversial) appear to disfavour bulk P and T violation. Hence, the latest theoretical works [32] [33] have concentrated on studying the effects of layering on anyon superconductivity, in order to understand whether P and T violation survive layering. However, even if present day high T_c superconductors are unlikely candidates of anyon superconductors, efforts to understand many anyon systems and, in particular, their superconducting behaviour remains undiminished.

Problems

1. The standard mean field approach to anyon superconductivity starts from fermions in a magnetic field. But if we start with anyons as bosons plus attached flux-tubes, then the mean field problem to be solved is that of bosons (albeit hard-core because of the hard-core nature of anyons) in a magnetic field. Can you solve this problem? (This is as yet unsolved, but for starters, look up Ref.[34].)

5. Anyons in Field Theory

Bosons and fermions are introduced in a second quantised field theory formalism through field operators that obey local commutation or anticommutation rules. However, anyon field operators , being representations of the braid group instead of the permutation group, cannot obey local commutation rules. This makes it hard to construct anyon field theories in the canonical way. Path integral quantisation is no easier, since the phase picked up by any anyon moving along a particular path depends on the position of all the other anyons in the system. Hence, unlike the case with bosons or fermions, paths cannot be weighted by a unique weight factor. But the study of anyons in a field theoretic formulation was rendered possible by the introduction [10] of an elegant idea called the Chern-Simons construction. Here, anyons were introduced as interacting fermions or bosons. It is this formulation (as opposed to more abstract formulations [35]) that we shall study in this section.

Let us consider any field theory in $2+1$ dimensions described by a Lagrangian L and having a conserved current j_μ - *i.e.*, $\partial^\mu j_\mu = 0$. We can manufacture a gauge field a_μ and add to the Lagrangian

$$\Delta L = j_\mu a^\mu - \frac{\mu}{2} \epsilon_{\mu\nu\alpha} a^\mu \partial^\nu a^\alpha. \tag{5.1}$$

Is this an allowed extension of the Lagrangian? Firstly, if a_μ is to be a gauge field, ΔL has to be gauge invariant. Under $a_\mu \to a_\mu - \partial_\mu \Lambda$,

$$\int \Delta L \, d^3x \to \int \Delta L \, d^3x - \int j_\mu (\partial^\mu \Lambda) \, d^3x + \frac{\mu}{2} \int \epsilon_{\mu\nu\alpha} (\partial^\mu \Lambda)(\partial^\nu a^\alpha) \, d^3x. \tag{5.2}$$

The third term obviously vanishes (upto surface terms) on integrating by parts due to the anti-symmetry of $\epsilon_{\mu\nu\alpha}$. The second term also certainly vanishes on integrating by parts at the equation of motion level, since $\partial_\mu j^\mu = 0$. However, for any explicit current, we can, in fact, construct a gauge invariant coupling of the current with a local gauge field. For example, when j_μ is a fermionic current, the gauge invariant coupling is just $j_\mu a^\mu = \bar{\psi}\gamma_\mu \psi a^\mu$. But when j_μ is a scalar current, $j_\mu a^\mu$ is replaced by $D_\mu \phi D^\mu \phi$. Hence, the ΔL in Eq.(5.1) is certainly an allowed gauge invariant extension of any Lagrangian.

The term $(\Delta L)_{CS} = (\mu/2)\epsilon_{\mu\nu\alpha}a^\mu\partial^\nu a^\alpha$ is called the Chern-Simons (CS) term. The CS term and its non-abelian generalisations are interesting field theories in their own right and have been studied for years [36] in other contexts. More recently, they have shot into prominence as prototypes of 'topological' or 'metric independent' field theories [37]. However, for our purposes here, we shall only recollect the following salient features of Chern-Simons theories. An abelian CS field theory is described by

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mu}{2}\epsilon_{\mu\nu\alpha}A^\mu\partial^\nu A^\alpha, \tag{5.3}$$

where $F^{0i} \equiv \partial_0 A_i - \partial_i A_0 = E_i$ and $(1/2)\epsilon_{0ij}F^{ij} = B$. It is clear that the electric field E_i is a two component vector and the magnetic field B is a pseudoscalar, and that these are the only non-zero components of $F^{\mu\nu}$, since we are in $2+1$ dimensions. The parameter μ in Eq.(5.3) has the dimensions of a mass and is the gauge invariant mass term for the gauge field. This can be seen by explicitly computing the propagator. Furthermore, the CS term is odd under the discrete symmetries parity P $(x \to -x, y \to y)$ and time reversal T $(t \to -t)$. Thus, the Lagrangian in Eq.(5.3) describes a $U(1)$ gauge theory with a massive photon. A gauge invariant mass for the photon has been introduced at the expense of the violation of P and T.

With this introduction to CS or topological field theories, let us get back to the study of Eq.(5.1). Notice that the Lagrangian in Eq.(5.1) does not include the usual kinetic piece, the $F_{\mu\mu}F^{\mu\nu}$ term for the gauge field. Also, from the equation of motion, we get

$$j_\mu = \mu\epsilon_{\mu\nu\alpha}\partial^\nu a^\alpha, \tag{5.4}$$

which, when coupled with a gauge condition allows for a solution for the gauge field in terms of the current j_μ. (We shall see this explicitly later in this section.) Hence, the motion of the gauge field a_μ is completely determined by the current j_μ and has no independent dynamics. In this respect, CS theories of relevance to anyons (or metric-independent field theories) differ from most of the CS-models studied earlier. However, the gauge field does affect the statistics of the current carrying particles. Integrating the zeroth component of

Eq.(5.4) over all space, we get

$$\begin{aligned} \int j^0 \, d^2\mathbf{r} &= \mu \int \epsilon_{0ij} (\partial^i a^j) \, d^2\mathbf{r} \\ \Rightarrow \quad q &= \mu\phi. \end{aligned} \tag{5.5}$$

Thus, every charge q of the current j_μ is accompanied by a flux ϕ and is an anyon. This mechanism of attaching fluxes to charges is called the CS construction. The current j_μ can be a fermion number current ($j_\mu = \bar{\psi}\gamma_\mu\psi$) in which case fermions turn into anyons, or a bosonic current ($j_\mu = \phi^\dagger \partial_\mu \phi - (\partial_\mu \phi)^\dagger \phi$) so that bosons turn into anyons, or even a topological current so that topological objects like solitons and vortices turn into anyons.

Let us compute the statistics of these particles in the field theory context. In the path integral formalism, when two such objects (flux carrying charges) are exchanged, the phase is given by $e^{iS_{\text{ex}}}$ where S_{ex} is the action involved in exchanging them.

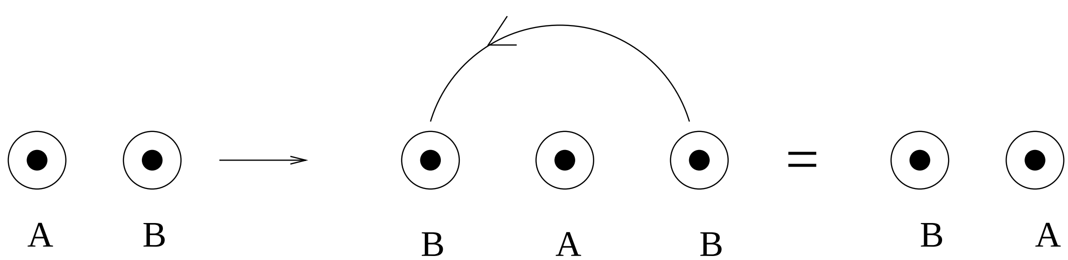


Fig. 17

Now, it is clear that $S_{\text{ex}} = (1/2)S_{\text{rot}}$, where S_{rot} is the action for taking particle B all around particle A.

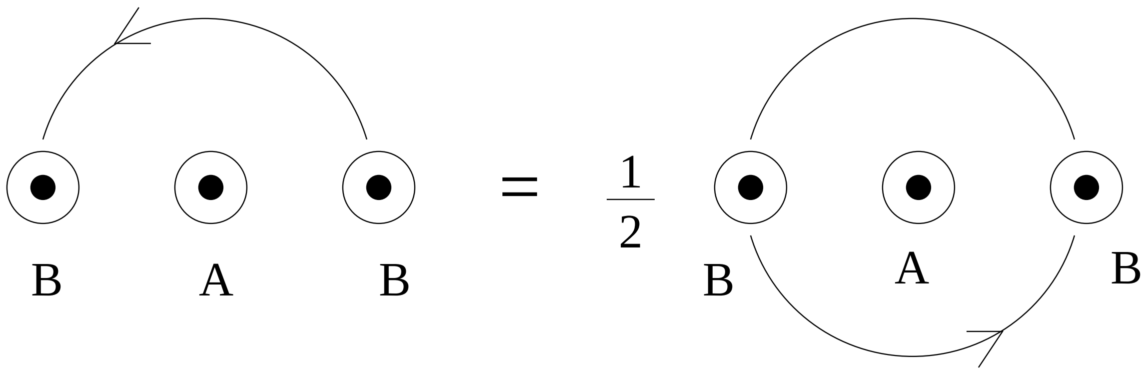


Fig. 18

$S_{\rm rot}$, in turn, is twice the Aharanov-Bohm action $S_{\rm AB}$ required to take a single charge around a single flux, because $S_{\rm rot}$ involves taking both a charge and a flux around a charge and a flux. Therefore,

$$S_{\rm ex} = S_{\rm rot}/2 = S_{\rm AB}. \tag{5.6}$$

The Aharanov-Bohm phase is easily computed since we know the 'electromagnetic' Lagrangian (Eq.(5.1)) that governs the motion of a charged particle in the field of a flux-tube and is given by

$$\begin{aligned} S_{\rm ex} = S_{\rm AB} &= \int \Delta L \, dt \\ &= \int j_i a^i \, dt + \frac{\mu}{2} \int \epsilon_{0ij} a_i \partial_0 a_j \, dt \end{aligned} \tag{5.7}$$

(in the $a_0 = 0$ gauge). Since $j_i = qv_i$, the first term in Eq.(5.7) gives

$$\int j_i a^i \, dt = q \int \mathbf{a} \cdot d\mathbf{l} = q\phi. \tag{5.8}$$

Also, from the equation of motion in Eq.(5.4), we see that the motion of the gauge field is related to the current as

$$\mu \epsilon_{0ij} \partial^0 a^j = -j_i. \tag{5.9}$$

Substituting Eq.(5.9) in Eq.(5.7), we see that

$$S_{\rm ex} = q\phi/2 \equiv \alpha. \tag{5.10}$$

Thus, the statistics of a charge with a flux induced on it by a CS construction is $\alpha = q\phi/2$,

consistent with what we had earlier derived for the charge-flux composite in the Hamiltonian formulation.

The relation between the Lagrangian CS field theory formulation and the Hamiltonian formulation for many anyons introduced in Sec.(4), can be explicitly demonstrated [6] by carrying out the canonical transformation between the Lagrangian and Hamiltonian formulations for a gas of anyons. We start with the Lagrangian formulation of a gas of anyons (represented by point particle bosonic charges with attached flux-tubes) given by

$$L = \sum_\alpha [\frac{m}{2}\dot{\mathbf{r}}_\alpha^2 + q a_0(\mathbf{r}_\alpha) + q\dot{\mathbf{r}}_\alpha \cdot \mathbf{a}(\mathbf{r}_\alpha)] - \frac{\mu}{2}\int \epsilon_{\mu\nu\alpha} a^\mu \partial^\nu a^\alpha \, d^2\mathbf{r} \tag{5.11}$$

where α is the particle index and the coupling between the particle with charge q and the gauge field a_μ is as in standard electromagnetism. However, the usual kinetic term of electromagnetism $F_{\mu\nu}F^{\mu\nu}$ is now replaced by the CS term, in accordance with Eq.(5.1). This Lagrangian may be rewritten as

$$L = \sum_\alpha \frac{m}{2}\dot{\mathbf{r}}_\alpha^2 + \int a_0(j_0 - \mu\epsilon_{0ij}\partial_i a_j)\, d^2\mathbf{r} + q\sum_\alpha \dot{\mathbf{r}}_\alpha \cdot \mathbf{a} + \frac{\mu}{2}\int \epsilon_{0ij} a_i \dot{a}_j \, d^2\mathbf{r} \tag{5.12}$$

where

$$j_0 = q\sum_\alpha \delta^2(\mathbf{r} - \mathbf{r}_\alpha) \tag{5.13}$$

represents the point particle charge density of the anyons. Notice that apart from the first term, all the terms are linear in a_0 or time derivatives. Also, since there is no kinetic term for a_0, the equation of motion with respect to a_0 yields the constraint

$$\frac{\partial L}{\partial a_0} = 0 \Rightarrow j_0 = \mu\epsilon_{0ij}\partial_i a_j = \mu b, \tag{5.14}$$

so that the field strength f_{ij} is completely determined by j_0. But for an abelian gauge field theory, the entire gauge invariant content of the gauge field a_i is contained in the field strength f_{ij}. Hence, if we eliminate the extra degrees of freedom in a_i by a gauge condition, we can actually solve for a_i in terms of j_0, so that the motion of the gauge field is entirely determined by the fields forming j_0 and has no independent dynamics. The CS gauge field has been introduced merely to attach flux-tubes to charges and turn them into anyons.

To go to the Hamiltonian formulation, we require the canonical momenta given by

$$\mathbf{p}_\alpha = \frac{\partial L}{\partial \dot{\mathbf{r}}_\alpha} = m\dot{\mathbf{r}}_\alpha + q\mathbf{a} \tag{5.15}$$

and

$$p_i = \frac{\partial L}{\partial \dot{a}_i} = -\frac{\mu}{2}\int \epsilon_{0ij} a^j \, d^2\mathbf{r}. \tag{5.16}$$

The Hamiltonian can now be written as

$$\begin{aligned}
H &= \sum_\alpha \mathbf{p}_\alpha \dot{\mathbf{r}}_\alpha + p_i \dot{a}_i - L \\
&= \sum_\alpha (m\dot{\mathbf{r}}_\alpha + q\mathbf{a}) \cdot \dot{\mathbf{r}}_\alpha - \frac{\mu}{2}\int \epsilon_{0ij} \dot{\mathbf{a}}_i \mathbf{a}_j \, d^2\mathbf{r} - \sum_\alpha \frac{m}{2}\dot{\mathbf{r}}_\alpha^2 - q\sum_\alpha \dot{\mathbf{r}}_\alpha \cdot \mathbf{a} + \frac{\mu}{2}\int \epsilon_{0ij}\dot{\mathbf{a}}_i \mathbf{a}_j \, d^2\mathbf{r} \\
&= \sum_\alpha \frac{m}{2}\dot{\mathbf{r}}_\alpha^2 = \sum_\alpha \frac{(\mathbf{p}_\alpha - q\mathbf{a})^2}{2m},
\end{aligned} \tag{5.17}$$

which, in terms of the velocity is just the Hamiltonian of free particles. So the classical equations of motion are identical to the equations of motion for free particles. What has been altered, however, is the relation between canonical velocity and momenta (see Eq.(5.16)). Hence, the quantum commutation relations are no longer the same. This is the significance of introducing the CS term without the usual kinetic piece and is consistent with our original introduction of a_μ in Sec.(4), as just a way of enforcing anyon boundary conditions in a different gauge.

Let us now solve for a_i in terms of j_0 from Eq.(5.14) and the additional gauge condition

$$\partial_i a^i = 0. \tag{5.18}$$

The solution to these two equations, for the j_0 given in Eq.(5.13), is given by

$$\begin{aligned}
a_i(\mathbf{r}) &= \frac{1}{2\pi\mu}\int d^2\mathbf{r}' \epsilon_{0ij} \frac{(\mathbf{r}-\mathbf{r}')_j}{|\mathbf{r}-\mathbf{r}'|^2} j_0(\mathbf{r}') \\
&= \frac{q}{2\pi\mu}\sum_\alpha \epsilon_{0ij}\frac{(\mathbf{r}-\mathbf{r}'_\alpha)}{|\mathbf{r}-\mathbf{r}'_\alpha|^2}
\end{aligned} \tag{5.19}$$

which can be easily checked. The gauge condition is obviously satisfied due to the anti-symmetry of ϵ_{0ij}. Also, by regularising the denominator of a_i as explained in Sec.(4), we

can explicitly check that

$$b = \nabla \times \mathbf{a} = \frac{1}{2\pi\mu} \int j_0(\mathbf{r}') 2\pi \delta(\mathbf{r} - \mathbf{r}')\, d^2\mathbf{r}' = \frac{j_0(\mathbf{r})}{\mu}. \tag{5.20}$$

Notice that the many body Hamiltonian in Eq.(5.17), along with the solution for a_i in Eq.(5.20), is precisely the many body Hamiltonian (Eq.(4.1)) and gauge field a_i (Eq.(4.2)) that was used in Sec.(4), where it was obtained by generalising the two anyon Hamiltonian. Thus, we have established the equivalence of the Lagrangian CS formulation and the many body Hamiltonian formulation used in the earlier sections [6].

Let us now study a study a specific example of a field theory - the abelian Higgs model with a CS term [12] - whose solitons (classical solutions of the equations of motion) are 'anyonic'. We shall first construct the topologically non-trivial vortex solutions of the abelian Higgs model and then show that they are charged and have fractional spin in the presence of a CS term. The Lagrangian for this model is given by

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial_\mu - iqA_\mu)\phi^* (\partial^\mu + iqA^\mu)\phi - c_4 (\phi^*\phi - \frac{c_2}{2c_4})^2 + \frac{\mu}{4} \epsilon_{\mu\nu\alpha} F^{\mu\nu} A^\alpha. \tag{5.21}$$

This model has a $U(1)$ gauge symmetry. However, when $c_2 > 0$, the potential energy $V(\phi)$ is minimised when

$$V(\phi) = c_4 (\phi^*\phi - \frac{c_2}{2c_4})^2 = 0 \Rightarrow |\phi| = \sqrt{\frac{c_2}{2c_4}} = v, \tag{5.22}$$

where v is the vacuum expectation value. Hence, the $U(1)$ symmetry is spontaneously broken by the vacuum.

The usual vacuum has

$$\phi(r,\theta) = v, \mathbf{A}(r,\theta) = 0 \quad \text{and} \quad A_0(r,\theta) = 0, \tag{5.23}$$

which is a solution of the equations of motion and perturbation theory is built up by expanding the fields in modes around this vacuum. But besides the vacuum solution, this theory also possesses topologically non-trivial finite energy solutions [38]. To have finite

energy solutions, all we need to ascertain is that

$$|\phi(r \to \infty, \theta)| = v, \mathbf{A}(r \to \infty, \theta) = 0 \quad \text{and} \quad A_0(r \to \infty, \theta) = 0, \tag{5.24}$$

so that the energy integral does not diverge. However, the condition for the scalar field is satisfied even if the scalar field has a non-trivial phase at infinity. So a solution of the equations of motion which satisfies

$$\begin{aligned} \phi(r \to \infty, \theta) &= ve^{in\theta} \\ \Rightarrow \partial_\theta \phi((r \to \infty, \theta)) &= \frac{1}{r}\frac{\partial}{\partial \theta} ve^{in\theta} = i\frac{n}{r} \end{aligned} \tag{5.25}$$

has finite potential energy. Finiteness of the kinetic energy of the scalar field also implies that

$$\int (\partial_i - iqA_i)\phi^*(\partial^i - iqA^i)\phi \; d^2\mathbf{r} = \text{finite}, \tag{5.26}$$

which, in turn, gives the conditions on the asymptotic behaviour of $\mathbf{A}$ and A_0 as

$$A_\theta(r \to \infty, \theta) = \frac{n}{qr}, A_r = 0 \quad \text{and} \quad A_0 = 0. \tag{5.27}$$

Solutions that satisfy Eq.(5.25) and Eq.(5.26) are topologically non-trivial vortices. They are called vortex solutions because they carry flux - *i.e.*, using these solutions, we see that the flux is given by

$$\int B \, d^2\mathbf{x} = \int \mathbf{A} \cdot d\mathbf{l} = \int (\frac{n}{qr}) r d\theta = \frac{2\pi n}{q}, \tag{5.28}$$

where n is called the vorticity.

These solutions are topologically stable because solutions with different values of n are not deformable into each other without changing the scalar field configuration throughout the infinite boundary of space. This argument is easily understood pictorially. In Fig.(19), we have the vacuum solution with $\phi = v$ everywhere, including at spatial infinity,

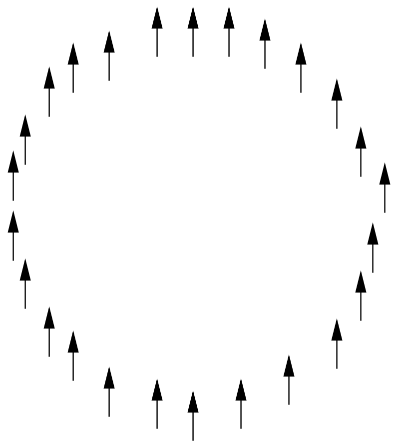


Fig. 19

whereas in Fig.(20), we have the one-vortex solution with $\phi(\infty, \theta) = ve^{i\theta}$.

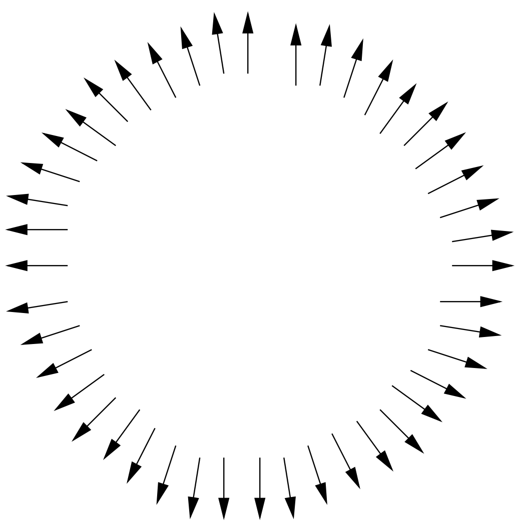


Fig. 20

Both the vacuum and the one-vortex configurations satisfy $V(\phi) = 0$. But to deform one solution to another, we need to go through configurations where $\phi(\infty, \theta)$ is neither v nor $ve^{i\theta}$ and which have $V(\phi) \neq 0$. Since this has to occur throughout the infinite boundary of two dimensional space, this will cost an infinite energy. In general, from Eq.(5.21), we see that the vacuum expectation value v takes values on a circle (since ϕ is complex and

$\phi^2 = v^2$). Hence, the configuration space of the vacuum is denoted by S^1_c. The boundary of real (2-dimensional) space is also a circle and is denoted by S^1_r. Hence, the boundary condition expressed in Eq.(5.25) denotes a mapping from S^1_r to S^1_c and the vorticity n represents the number of times ϕ encircles S^1_c when $\mathbf{r}$ encircles S^1_r once. Each of these solutions falls in a different topological class and is stable, since it requires an infinite energy to change the configuration of ϕ throughout the infinite boundary.

Now, let us introduce a CS term for the gauge field. We had earlier seen that in the presence of the CS term, the flux and charge get related. Hence, now, these vortex solutions also possess a charge given by

$$Q = \mu\phi = \mu(\frac{2\pi n}{q}). \tag{5.29}$$

Since these charged vortices carry both charge and flux, by our earlier arguments in Sec.(1), they are anyons with spin $j = Q\phi/4\pi = \pi n^2\mu/q^2$ and statistics phase $\alpha = Q\phi/2 = 2\pi^2 n^2/q^2$. Notice that here an explicit kinetic term for the gauge field has been included, so that the gauge field is really a dynamical degree of freedom. Hence, this model is not really relevant to anyons. However, even when the usual kinetic piece is switched off, the model continues to exhibit anyonic solutions [39].

This was just one example of a field theory whose solitonic excitations are anyons. Another example is the $O(3)$ σ-model with the Hopf term, whose skyrmionic excitations are anyons [28]. In all such models, the basic ingredient is the CS term (or equivalently the Hopf term) which relates the charge and the flux.

The CS term can also be extended to non-abelian theories with the appropriate Lagrangian being given by [36]

$$L = \frac{1}{2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu} - \frac{\mu}{2}\epsilon_{\mu\nu\alpha}\mathrm{tr}(F^{\mu\nu}A^\alpha - \frac{2}{3}gA^\mu A^\nu A^\alpha). \tag{5.30}$$

This Lagrangian can be shown to be gauge invariant provided $4\pi\mu/g^2 =$ integer. As in the abelian theory, μ is the gauge invariant mass for the gauge boson and the CS term is odd under parity and time reversal. The pure non-abelian CS Lagrangian without the usual kinetic piece has been recently related to problems in topology and knot theory,

integrable models in statistical mechanics, conformal field theories in 1+1 dimensions and 2+1 dimensional quantum gravity. (For references to the original papers, see Ref.[28].) The remarkable feature of pure CS theories that makes it amenable to exact solutions is its general covariance, — the Lagrangian is not only Lorentz invariant, but it is also generally covariant without any metric insertions. Hence, correlation functions of the pure CS theory depend only on the topology of the manifold and not on details such as the metric on the manifold. Besides theoretical interest in non-abelian theories, there have also been speculations [40] that non-abelions, — generalisations of anyons that form non-abelian representations of the braid group, — may play a role in the even denominator FQHE. However, since non-abelian CS theories are not directly related to anyons, we shall not pursue this topic any further here.

Problems

1. Similar to vortices and charged vortices in $2+1$ dimensions, in $3+1$ dimensions we have monopoles and dyons. The Georgi-Glashow model in $3+1$ dimensions, which consists of an $SO(3)$ gauge field A_μ^a interacting with an isovector Higgs field $\vec{\phi}$ is one of the simplest examples. The Lagrangian for this model is given by

$$L = -\frac{1}{4}\mathrm{tr}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mathrm{tr}\ D_\mu\phi D^\mu\phi - \frac{\lambda}{4}(\vec{\phi}^2 - a^2)^2,$$

where $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g\epsilon_{abc}A_\mu^b A_\nu^c$ and $(D_\mu\phi)^a = \partial_\mu\phi^a - g\epsilon_{abc}A_\mu^b\phi^c$. Can you construct monopole solutions for this field theory by analogy with the vortex solution in Eqs. (5.25) and (5.26)?

b. Why are monopoles topologically stable?

c. Can you guess the generalisation of the Chern-Simons term that induces charge on the monopoles (converting them to dyons)?
Some useful references are Refs.[41] and [42].

6. Anyons in the Fractional Quantum Hall Effect

Any study of anyons would be incomplete without an account of its most outstanding success — its application to the FQHE. This is the only physical system where there exists incontrovertible evidence for the existence of anyons, because quasi-particle excitations over the FQHE ground state have been explicitly shown to obey fractional statistics. However, since familiarity with the Quantum Hall system is not in the repertoire of the average graduate students, we shall introduce the background material in some detail.

Let us first remind ourselves of the Hall Effect. Here, electrons in a plane show transverse conductivity when a magnetic field is applied perpendicular to the plane. The Hall geometry is depicted in Fig.(21).

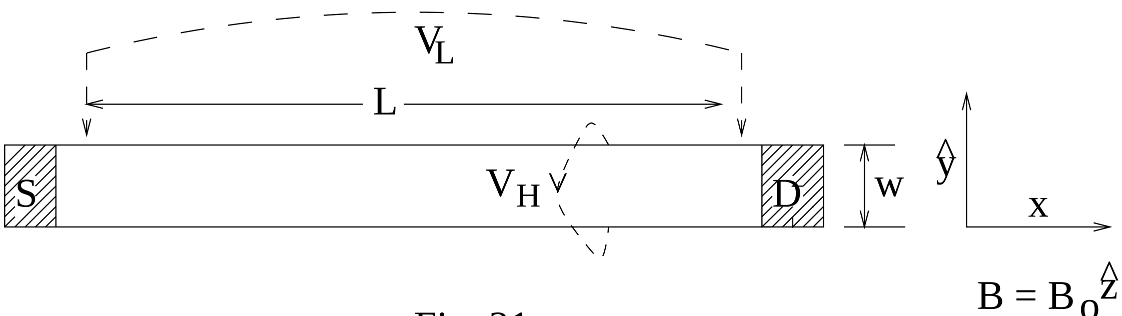


Fig. 21

Electrons are allowed to move from a source S to a drain D causing a current I, which is measured, as are the longitudinal and transverse voltage drops V_L and V_H. The existence of a non-zero V_H can be explained just by classical electrodynamics using the equation

$$\mathbf{F} = e\mathbf{E} + e\mathbf{v} \times \mathbf{B} \tag{6.1}$$

for the electrons with charge $q = e$. For the geometry in Fig.(21), we have the equations

$$\dot{v}_x = \frac{eE_x}{m} + \frac{ev_y B_0}{m}, \quad \dot{v}_y = -\frac{ev_x B_0}{m} \tag{6.2}$$

whose solution is

$$v_x = v_0 e^{i\omega t}, \quad v_y = -\frac{E_x}{B_0} + iv_0 e^{i\omega t} \tag{6.3}$$

where v_0 is the initial velocity and $\omega = eB_0/m$ is the frequency of the cyclotron motion.

The constant term in v_y represents the drift velocity $\mathbf{v}_d$ and the Hall current is given by

$$\mathbf{j}_H = \rho e \mathbf{v}_d \Rightarrow j_{Hx} = 0, j_{Hy} = -\frac{\rho e E_x}{B_0}, \tag{6.4}$$

where ρ is the density of charge carriers. By changing the Hall geometry, we can also find

$$j_{Hx} = \frac{\rho e E_y}{B_0}, \quad j_{Hy} = 0, \tag{6.5}$$

when the electric field is along the y-direction. Hence, if we define a conductivity matrix σ_{ij} by

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \tag{6.6}$$

then classically, $\sigma_{xy} = -\sigma_{yx} = -\rho e/B_0$ and $\sigma_{xx} = \sigma_{yy} = 0$. The resistivity matrix $\rho_{ij} = \sigma_{ij}^{-1}$ and Eq.(6.6) leads to

$$\rho_{xy} = -\rho_{yx} = \frac{B_0}{e\rho} \quad \text{and} \quad \rho_{xx} = \rho_{yy} = 0. \tag{6.7}$$

We have not taken collisions between electrons into account in deriving the above classical result. However, a more realistic picture requires collisions and by incorporating them, we get the semiclassical equations of motion given by

$$\begin{aligned} & < \dot{v}_x > = \frac{eE_x}{m} + \frac{q < v_y > B_0}{m} - \frac{< v_x >}{\tau} \\ \text{and} \qquad & < \dot{v}_y > = -\frac{e < v_x > b_0}{m} - \frac{< v_y >}{\tau}, \end{aligned} \tag{6.8}$$

where $< \mathbf{v} >$ is the average velocity of the electrons and τ is the average time between collisions. In equilibrium, $< \dot{\mathbf{v}} >= 0$ and we get

$$v_x = \frac{\tau e/m}{(\omega^2\tau^2 + 1)} E_x, \quad v_y = -\frac{\omega\tau^2 e/m}{(\omega^2\tau^2 + 1)} E_x \tag{6.9}$$

where, as before, $\omega = eB_0/m$. Note that unlike Eq.(6.5) which does not give the correct $B_0 \to 0$ limit, (Hall current goes to infinity instead of vanishing), here $\omega \to 0$ as $B_0 \to 0$ and

the Hall current vanishes. Thus, in the semiclassical limit, the elements of the conductivity matrix are

$$\sigma_{xx} = \sigma_{yy} = \frac{\rho e^2 \tau / m}{(\omega^2 \tau^2 + 1)} \quad \text{and} \quad \sigma_{yx} = -\sigma_{xy} = -\frac{\rho e^2 \tau^2 \omega / m}{(\omega^2 \tau^2 + 1)} \tag{6.10}$$

and the elements of the resistivity matrix are

$$\rho_{xx} = \rho_{yy} = \frac{m}{\rho e^2 \tau} \quad \text{and} \quad \rho_{yx} = -\rho_{xy} = -\frac{m\omega}{\rho e^2} = \frac{B_0}{e\rho}. \tag{6.11}$$

Notice that the off-diagonal elements of the resisitivity are unchanged from their classical values.

Now, when the Hall resistance was experimentally measured [43] in 1980, at low temperatures (0 - 2 deg K) and in strong magnetic fields (1 - 20 *Tesla*), a surprising result was found. σ_H showed plateaux (as shown schematically in Fig.22) instead of varying linearly with $1/B_0$ as expected classically or semiclassically.

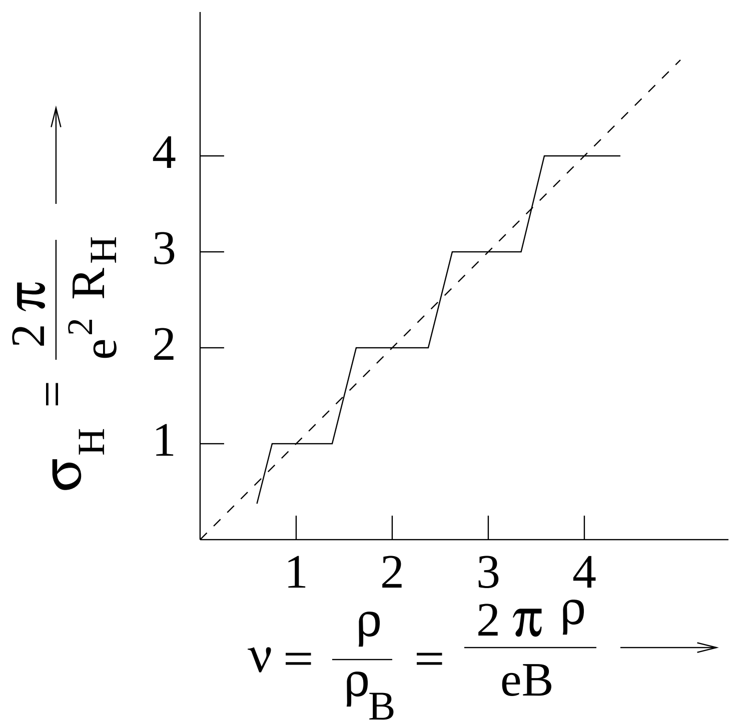


Fig. 22

Furthermore, the longitudinal resistance, instead of being constant as expected by Eq.(6.10), vanished at the plateaux and peaked in between as shown (schematically) in Fig.23.

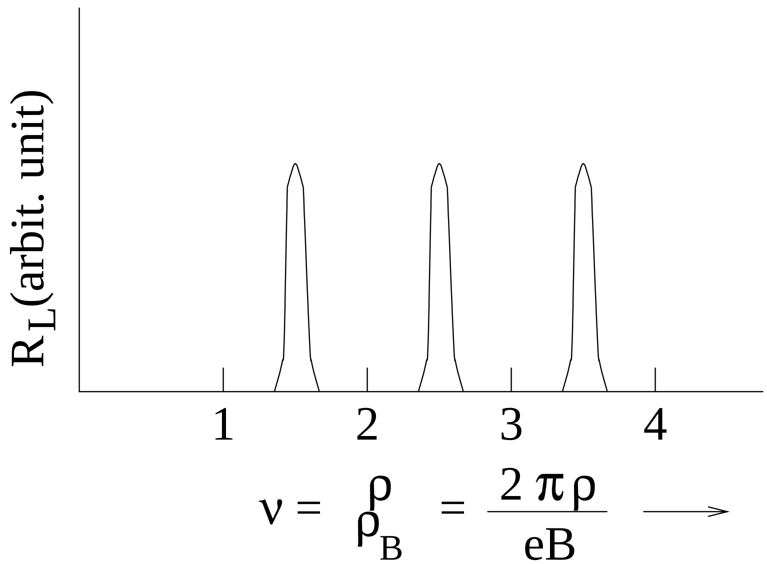


Fig. 23

Even more surprisingly, though the behaviour of the Hall resistance at the edges of the steps was non-universal, the R_H at the plateaux was unexpectedly constant and reproducible to the accuracy of one part in 10^5. This effect was called the Integer Quantum Hall Effect (IQHE), because the midpoints of the plateaux occured at integer values of a ratio $\nu = \rho/\rho_B$ (where $\rho_B = eB_0/2\pi$). Since ρ_B is the degeneracy of each Landau level, this ratio which expressed the fraction of filled Landau levels was called the filling factor. The filling factor could either be thought of as a measure of B_0 at fixed ρ or as a measure of ρ at fixed B_0.

The remarkable accuracy of the experimentally measured resistances is explained by the following observation. Usually the resistances $R_H = V_H/I$ and $R_L = V_L/I$ are related to the resistivities by geometric factors of length or width. However, for the Hall geometry, when $L >> W$ and the voltage drops are measured sufficiently far from the actual edges so that the applied current density is uniform, we see that

$$R_H = \frac{V_H}{I} = \frac{E_y W}{j_x W} = \rho_{xy}. \tag{6.12}$$

Hence, geometric factors which can never be measured to accuracies of one in 10^5 have

cancelled out leaving the transverse resistance equal to the transverse resistivity. In fact, the quantisation of the experimentally measured Hall resistance is so accurate, that the Quantum Hall system is now used for the most precise determination of the fine structure constant.

The midpoint values of the plateaux can be easily understood by studying the quantum mechanical problem of non-interacting electrons in a transverse magnetic field. In Sec.(4), we studied the problem of an electron in a transverse magnetic field and found that the energy levels are given by

$$E_n = (n + 1/2)\frac{eB_0}{m}$$

with the degeneracy

$$\mathrm{Deg} = \rho_B = \frac{eB_0}{2\pi},$$

where we have substituted $\omega = eB_0/m$ in Eqs.(4.14) and (4.18). Hence, when the filling factor $\nu = \rho/\rho_B$ is an integer, it is clear that precisely an integer number of Landau levels are filled. As mentioned in Sec.(4), this implies a gap to single particle excitations, given by $\omega = eB_0/m$. Also, unlike the case in Sec.(4), here B_0 is independent of ρ and there is no argument for a massless collective excitation. On the contrary, explicit calculations show that the collective excitation is massive. So the system is particularly stable when the density (or equivalently the magnetic field) is such that ν is an integer. From Eq.(6.11), we see that for these values of the density,

$$R_H = \frac{B_0}{e\rho} = \frac{B_0}{e\rho_B}\frac{1}{\text{integer}} = \frac{2\pi}{e^2}\frac{1}{\text{integer}}, \tag{6.13}$$

in accordance with the experimental values of R_H at the midpoint of the plateaux. Thus, the densities of the electrons at the midpoint of the plateaux have been identified with fully filled Landau levels and the correct Hall conductivity is predicted for these densities.

However, to understand why the conductivity remains fixed even when B_0 (or ρ) is changed from the midpoint value is harder. At a hand-waving level, we can argue that because of the stability of the system when ν is an integer, even when the field B_0 is changed slightly, the system prefers to keep the average density fixed such that ν is an

integer and accomodate the deviation in B_0 as a local fluctuation. These local fluctuations do not contribute to the conductivity because they get 'pinned' or 'localised' by impurities in the sample. Hence, the conductivity stays fixed at the value that it had for $\nu =$ integer. At a slightly more rigorous level, the idea is that weak disorder in the system (due to impurities and imperfections in the sample) leads to the formation of some localised states , whereas other states are extended. Current can only be carried by extended states. Hence, if the density of states as a function of the energy had the pattern shown in Fig.(24),

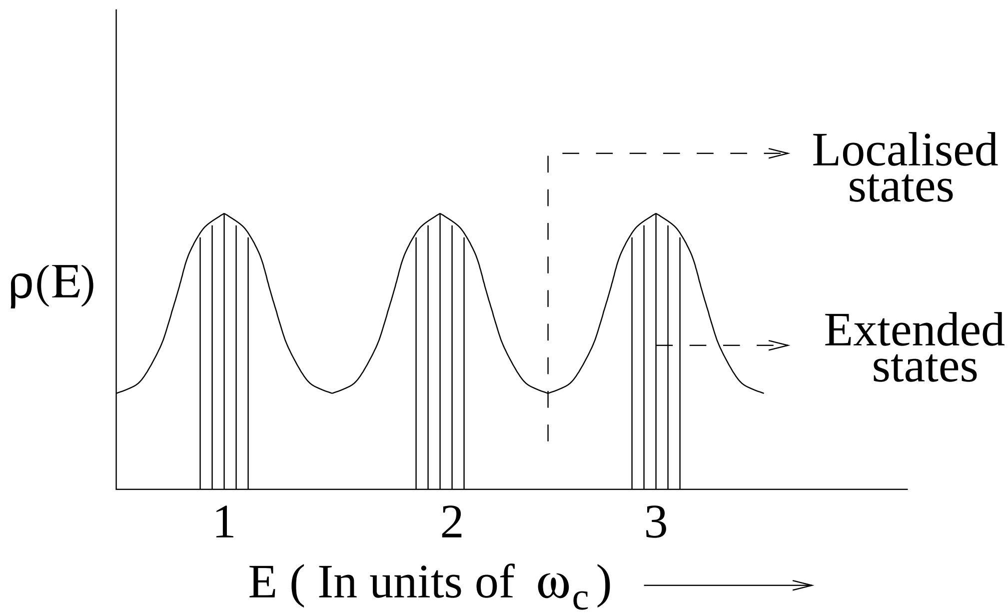


Fig. 24

then as the Fermi level spans each localised region, the current will remain constant (plateau in the Hall conductivity), while when it spans the extended states, the current (and hence, conductivity) will increase. To study in more detail the kinds of disorder potentials that can lead to the density of states diagram in Fig.(24) would lead us too far afield. For further details, we refer the reader to Ref.[44].

In 1984, in stronger magnetic fields and cleaner samples, the FQHE was seen [2], where the midpoints of the plateaux were found to occur at fractional values of the filling factor such as 1/3, 1/5, etc. This was unexpected because the single particle quantum mechanics

analysis that was used for the IQHE predicted that at $\nu =$ fraction, the system would be highly degenerate and not at all stable. So, the very existence of the effect showed that inter-electron interactions which were ignored in the earlier study must be important. Hence, the appropriate Hamiltonian for the FQHE is expected to be

$$H = \sum_i \frac{(p_i - eA_i)^2}{2m} + \sum_i V(\mathbf{r}_i) + \frac{1}{2}\sum_{i \neq j} V(\mathbf{r}_i - \mathbf{r}_j) \tag{6.14}$$

where $V(\mathbf{r}_i - \mathbf{r}_j)$ denotes the inter-electron Coulomb repulsion and $V(\mathbf{r}_i)$ is a neutralising background potential. Unlike the IQHE problem, this Hamiltonian cannot be solved exactly even in the absence of disorder.

There have been several approaches to the theoretical understanding of the FQHE, mainly because no theory has provided a complete picture yet. Here, we shall concentrate on two approaches, both of which appear to use anyon ideas in a fundamental way — the trial wavefunction approach and the Landau-Ginzburg-Chern-Simons field theory approach.

The trial wavefunction approach was initiated by Laughlin [3] who proposed that the ground state of the FQHE system is described by the wavefunction given by

$$\psi(z_1, z_2, ...z_N) = \prod_{i<j}^{N} (z_i - z_j)^m e^{-\Sigma_i |z_i|^2/4l^2} \tag{6.15}$$

for the fraction $\nu = 1/m$. Here $z_i = x_i + iy_i$ is the position of the i^{th} particle and m has to be an odd integer to satisfy the criterion that the particles are electrons. He arrived at this wavefunction from certain general principles such as *a.*) the wavefunction should be antisymmetric under exchange, *b.*) the wavefunction should comprise of single particle states in the lowest Landau level, *c.*) the wavefunction should be an eigenstate of total angular momentum, and d.) inspired guesswork. This ansatz for the ground state wavefunction has had enormous success, mainly because no other ansatz has been found with lower energy.

To understand the significance of this wavefunction, let us write

$$|\psi|^2 = e^{-\beta\phi} \tag{6.16}$$

so that

$$\phi = -\frac{2m}{\beta}\sum_{i<j}^{N} \ln\frac{|z_i - z_j|}{l} + \frac{1}{2\beta}\sum_{i}^{N}\frac{|z_i|^2}{l^2}. \tag{6.17}$$

By setting the fictitious temperature $\beta = 1/m$, we see that ϕ can be interpreted as the potential energy of a two dimensional gas of classical particles of charge 'm' repelling each other through a logarithmic interaction and being attracted to the origin by a uniform (opposite) charge density $\rho_U = 1/2\pi l^2$. This potential energy is minimised (*i.e.*, $|\psi|^2$ is maximised) by a uniform distribution of the charge 'm' particles and is 'electrically' neutral everywhere, when the average density of the charge 'm' particles is precisely equal to ρ_U, which in turn implies that the average density of the electrons ρ is equal to $1/2\pi m l^2$. Hence, at these densities, the many electron wavefunction peaks when the coordinates z_i are uniformly distributed and is expected to be energetically favourable, since the Coulomb repulsion in Eq.(6.14) is minimised. For other values of ρ, the classical gas has excess charge 'm' particles either near the origin or near the sample boundary and hence fails to be uniformly distributed. So, the appropriate ψ peaks at a configuration of the z_i's that does not describe a uniform distribution of electrons and suffers from a high repulsive Coulomb energy. Hence, the Laughlin picture is that as the density is reduced from $\rho_B = eB/2\pi$ where one Landau level is filled, whenever $\rho = \rho_m = \rho_B/m \Rightarrow \nu = 1/m$, the energy is minimised and the state is stable.

The Laughlin state at $\nu = 1/m$ can be shown to have massive fractionally charged quasiparticle and quasihole single particle excitations, which obey anyon statistics. To see this, let us insert adiabatically one unit of flux through an imaginary thin solenoid piercing the many body state. The intermediate Hamiltonian as the flux is being evolved through the solenoid is given by

$$H = \sum_i \frac{(\mathbf{p}_i - e\mathbf{A}_i - e\mathbf{A}'_i)^2}{2m} + \sum_i V(\mathbf{r}_i) + \frac{1}{2}\sum_{i\neq j} V(\mathbf{r}_i - \mathbf{r}_j) \tag{6.18}$$

where $\mathbf{A}'_i$ is the gauge potential of the flux through the solenoid. As the flux changes, the wavefunction also changes so as to be an eigenstate of the Hamiltonian. At the end of the process, however, the Hamiltonian has returned to the original Hamiltonian, since one unit

of flux through an infinitely thin solenoid can be gauged away. (Remember that the term $e\mathbf{A}'_i$ in the Hamiltonian can be traded for a phase $e^{ie\phi}$ in the wavefunction in the 'anyon' gauge. But when the flux $\phi = \phi_0 = 2\pi/e$, the phase is just 1 and hence irrelevant.) But the state has not returned to the original state. As flux is adiabatically added, every single particle state

$$z^m e^{-|z|^2/4} \to z^{m+1} e^{-|z|^2/4} \tag{6.19}$$

-*i.e.*, its angular momentum increases by one. The state with the highest angular momentum moves over to the next Landau level and a new state appears at $m = 0$. If a single unit of flux is removed adiabatically, then

$$z^m e^{-|z|^2/4} \to z^{m-1} e^{-|z|^2/4}. \tag{6.20}$$

Hence, a state from the next Landau level moves down and the $m = 0$ state disappears. Thus, the effect of adding (or removing) one quantum of flux and then gauge transforming is to increase (or decrease) the angular momentum of the single particle states by one unit. But if we assume that the original state described by the Laughlin wavefunction is non-degenerate, (since no other states with the same energy have been found), then the new state, after evolution of the flux, has to describe an excited state of the original Hamiltonian, with a higher energy eigenvalue. This automatically proves that the quasiparticle or quasihole excitations have a gap.

The electric charge of these excitations can also be easily computed. Let us assume that the flux has been evolved through a very thin solenoid. Faraway from the solenoid, we expect that this state will be indistinguishable from the ground state, except that every level of the single particle states has moved over to the next level. Hence, if the flux point is surrounded by a large circle, then the charge that has entered or left the circle is just the average charge per state, provided the total charge and the total number of states are uniformly distributed. In position space, single particle states are labelled by the position vector $\mathbf{r}$ and are uniformly distributed with a degeneracy of $eB/2\pi$. By the plasma analogy, charge is also uniformly distributed in real space, with the density $e\rho_m = e^2B/2\pi m$ for $\nu = 1/m$. Thus, the charge per state is just $\pm e/m$ which is identified as the charge of the excitation.

The statistics of these quasiparticles and quasiholes can be explicitly found [45] by a Berry phase calculation. However, a much simpler way is to notice that these excitations have a flux $2\pi/e$ and a charge e/m and hence are anyons with fractional spin and statistics given by

$$j = \frac{q\phi}{4\pi} = \frac{e/m \times 2\pi/e}{4\pi} = \frac{1}{2m} \quad \text{and} \quad \alpha = \frac{q\phi}{2} = \frac{e/m \times 2\pi/e}{2} = \frac{\pi}{m}. \tag{6.21}$$

Eq.(6.19) implies that an explicit ansatz for the wavefunction of a quasihole excitation can be written as

$$\psi(z_0, z_1, ...z_N) = \prod_i (z_i - z_0) \prod_{i<j} (z_i - z_j)^m e^{-\Sigma_i |z_i|^2/4l^2} \tag{6.22}$$

where z_0 is the position of the infinitely thin solenoid. This ansatz is expected to be good except very near the solenoid. The wavefunction for the quasielectron involves derivative operators and is not as straighforward to understand, at least within the Laughlin scheme.

Laughlin's wavefunctions only explained the plateaux at the fractions $\nu = 1/m$ where m was an odd integer. But, experimentally, plateaux were seen at many other rational fractions $\nu = p/q$ where q was an odd integer. To explain the other fractions, the hierarchy scheme was evolved [4]. The idea was that quasiparticle or quasihole excitations over the Laughlin state themselves behaved like particles in a magnetic field and could form new correlated many body states which could represent the FQHE state at other fractions.

The hierarchy scheme is easily understood in the anyon language. Firstly, notice that the wavefunction for a quasihole excitation remains analytic (see Eq.(6.22)). Also, quasiholes obey anyonic statistics. Hence, the simplest possible wavefunction for a collection of many quasiholes of generic charge qe and statistics α' (using Laughlin's arguments to arrive at his wavefunction, but now remembering that the particles are anyons) is given by

$$\psi(z_{01}, z_{02}, ...z_{0M}) = \prod_{i<j}^{M} (z_{0i} - z_{0j})^{2k+\alpha'} e^{-(|q|/e)\Sigma_i |z_{0i}|^2/4l^2}. \tag{6.23}$$

This wavefunction looks exactly like Laughlin's wavefunction except that m is replaced by $2k + \alpha'$ to account for the changed statistics and in the exponent l^2 is replaced by

$el^2/|q|$ to account for the charge of the quasiholes which is qe. The same plasma analogy now suffices to find the density of quasiholes ρ_{qh} for which this wavefunction describes a uniformly distributed electrically neutral plasma and is hence likely to be an energetically favoured wavefunction. The plasma has 'charge' $(2k + \alpha')$ particles repelling each other and being attracted to the origin by a uniform 'charge' density $\rho = |q|/2\pi el^2$. So by our earlier argument, we shall have a uniform density of quasiholes when

$$\rho_{qh} = \frac{1}{(2k + \alpha')} \times \frac{|q|}{2\pi el^2}. \tag{6.24}$$

Now, we know that the charge of the quasiholes is $qe = -e/m$ and their statistics parameter $\alpha = \pi/m \Rightarrow \alpha' = 1/m$. (Remember that we are using complex notation in Eq.(6.23). Hence, under exchange of particles, $(z_{\mathrm{rel}})^{\alpha'} = (re^{i\theta})^{\alpha'} \rightarrow (re^{i(\theta+\pi)})^{\alpha'} = (z_{\mathrm{rel}})^{\alpha'} e^{i\pi\alpha'}$). Using this, the filling fraction of the quasiholes is given by

$$\nu_{qh} = \frac{\rho_{qh}}{\rho_B} = \frac{1}{m(2k + \frac{1}{m})}. \tag{6.25}$$

To find the equivalent density of electrons, notice that at a given density of electrons, the total charge remains fixed, whether it is counted as quasihole charges or electron charges. Hence, the total charge carried by the quasiholes is given by

$$q\rho_{qh} = -\frac{e}{m}\rho_{qh} = -\frac{e}{m^2(2k + \frac{1}{m})} \tag{6.26}$$

which, in turn, is equal to $e\rho_{eqh}$, where ρ_{eqh} is the equivalent density of electrons. So the total density of electrons and hence, the filling fraction is given by

$$\nu = \frac{1}{m} - \frac{1}{m^2(2k + \frac{1}{m})} = \frac{1}{m + \frac{1}{2k}}. \tag{6.27}$$

For $m = 3$ and $k = 1$, $\nu = 2/7$ and for $m = 3$ and $k = 2$, $\nu = 4/13$. Thus, we have obtained other fractions with odd denominators. This process can now be iterated. We can consider excitations over the $\nu = 2/7$ or the $\nu = 4/13$ state and form new correlated states with

those excitations. A little algebra shows that by repeating this procedure, we get filling fractions which can be written as

$$\nu = \cfrac{1}{m + \cfrac{1}{2k_1 + \cfrac{1}{2k_2 + \cdots + \cfrac{1}{2k_S}}}}. \tag{6.28}$$

Quasielectron excitations have a statistics factor $2k - \alpha$ instead of $2k + \alpha$ and have the same sign of charge as the electrons. Repeating the same analysis as above for quasielectron excitations gives ν as in Eq.(6.28) except that all the plus signs are replaced by minus signs. In general, when we allow for both quasiparticle and quasihole excitations over each state, the possible filling fractions are given by

$$\nu = \cfrac{1}{m + \cfrac{\alpha_1}{2k_1 + \cfrac{\alpha_2}{2k_2 + ... + \cfrac{\alpha_S}{2k_S}}}}. \tag{6.29}$$

where α_i are either +1 or -1 depending on whether quasiparticle or quasihole excitations are involved. All rational fractions with odd denominators are obtained once in this way. Also for the FQHE system, the hierarchy, as the above scheme to generate the fractions is called, works in the sense that for any fraction that has been observed, all the other fractions that lie before it in the hierarchy have also been observed.

The problem with the hierarchy scheme is that some fractions (*e.g.*, $\nu = 5/13$, at the third level of hierarchy) have not been observed. Also, for other fractions like $\nu = 6/13$ which is seen at the fifth level of hierarchy starting from the $\nu = 1/3$ state, the number of quasiparticles of various types is so much more than the number of original electrons, that it is hard to understand how the explanation of the original electrons forming a uniformly distributed $\nu = 1/3$ state survives. A way out of this predicament was suggested by Jain [13], who could directly obtain the wavefunctions for all the odd denominator fractions that have been seen. The starting point of his approach was to note that the FQHE was phenomenologically very similar to the IQHE, and hence, the theories of both the phenomena should also be related.

Let us start with an IQHE state, say, the p-filled Landau level state represented by

$$\nu = \frac{\rho}{\rho_B} = p \Rightarrow B_0 = \frac{2\pi\rho}{pe}, \tag{6.30}$$

and then attach fluxtubes of strength $\phi = 2k\phi_0$ to each electron. We know that this will convert fermions to anyons, but provided k is an integer, the statistical parameter $\alpha = e\phi/2 = 2k\pi$ is irrelevant (we have chosen the statistical charge to be the same as the real electric charge, which is always possible) and the electrons remain fermions. These fermions with attached fluxtubes are called 'composite fermions'. Now, within the mean field approach (valid for a high density of fermions), the flux of the fluxtubes can be spread out and we have ordinary fermions moving in an effective magnetic field

$$B_{\text{eff}} = \pm B_0 + 2k\phi_0 = \pm\frac{2\pi\rho}{pe} + \frac{4\pi k}{e}, \tag{6.31}$$

so that the filling factor ν for ordinary electrons is given by

$$\frac{1}{\nu} = \frac{\rho_B}{\rho} = \frac{eB_{\text{eff}}/2\pi}{\rho} = \pm\frac{1}{p} + 2k = \frac{2kp \pm 1}{p}. \tag{6.32}$$

Thus, the basic idea is that the FQHE for ordinary electrons occurs because of the IQHE for composite electrons — *i.e.*, the same kind of correlations between electrons are responsible both for the IQHE and the FQHE. When $p = 1$, this procedure yields the Laughlin fractions $\nu = 1/(2k \pm 1)$.

This idea can also be used to write down trial wavefunctions for arbitrary odd denominator fractions. For instance, to obtain the Laughlin wavefunctions, we start with the wavefunction for one filled Landau level given by

$$\psi_1(z_1, ...z_N) = \prod_{i<j}(z_i - z_j)e^{-\Sigma_i|z_i|^2/4l^2} \tag{6.33}$$

and then adiabatically introduce fluxtubes with $\phi = 2k\phi_0$ at the site of each electron. We know that adiabatic evolution of a flux unit at any point z_0 involves the factor $\Pi_i(z_i - z_0)$ from Eq.(6.22). Hence, insertion of $2k$ fluxtubes at the positions of all the electrons leads

to the wavefunction given by

$$\begin{aligned}\psi_{2k+1}(z_1, ...z_N) &= \prod_{i<j}(z_i - z_j)^{2k}\psi_1(z_1, ...z_N) \\ &= \prod_{i<j}(z_i - z_j)^{2k+1}e^{-\Sigma_i|z_i|^2/4l^2}\end{aligned} \tag{6.34}$$

which is precisely the Laughlin wavefunction for the filling fractions $\nu = 1/(2k+1)$. The same procedure can also be used to write down wavefunctions for the other fractions $\nu = p/(2kp+1)$ as

$$\psi_\nu = \prod_{i<j}(z_i - z_j)^{2k}\psi_p(z_1, ...z_N, \bar{z}_1, ...\bar{z}_N) \tag{6.35}$$

where $\psi_p(z_1, ...z_N, \bar{z}_1, ...\bar{z}_N)$ is the wavefunction for p filled Landau levels. But the form of the wavefunctions at $\nu = p/(2kp-1)$ is not as obvious. Explicit quasiparticle and quasihole wavefunctions can also be written down very easily in this formalism. Just as the FQHE wavefunctions are obtained from the IQHE wavefunctions ψ_p, the quasiparticle or quasihole wavefunctions for the FQHE are obtained from the quasihole wavefunctions ψ_p^+ or quasielectron wavefunctions ψ_p^- of the IQHE, by multiplying them by the factor $\Pi_{i<j}(z_i - z_j)^{2k}$. The quasihole wavefunction for the $\nu = 1/(2k+1)$ state in this formalism, coincides with the Laughlin quasihole for the same state, but the quasielectron wavefunction involves higher Landau levels and differs from Laughlin's ansatz and other trial wavefunctions in the literature. However, it appears much more natural and less arbitrary and works quite well numerically. The quasiparticle and quasihole charges and statistics can also be computed as was done for Laughlin's wavefunctions and the hierarchy wavefunctions and they coincide with the earlier results for any fraction $\nu = p/q$.

The wavefunction approach is a microscopic approach and has been fairly successful in explaining the phenomenon of the FQHE. However, it is not completely satisfactory, because it fails to illuminate all the symmetries and does not provide a complete understanding of the problem. To give an analogy, it is as if soon after superconductivity was discovered, the N-body projected BCS wavefunction in the coordinate representation was directly written down. Although it was correct, a complete understanding of the

phenomenon of superconductivity would not have been possible without discovering the Landau-Ginzburg theory and the phenomenon of Cooper pairing which actually led to the BCS theory. Hence, for the FQHE system too, there have been several attempts to find analogues of the Cooper pair that condenses and a Landau-Ginzburg theory. It is only in the last couple of years that these attempts have begun to bear fruit and an effective field theory called the Landau-Ginzburg-Chern-Simons (LGCS) theory has been evolved [14]. This approach uses anyon ideas and the Chern-Simons construction introduced in Sec.(5) to write down an effective action for the FQHE problem, whose saddle point solutions correspond to FQHE states. The key to this approach is the realisation that within the mean field approximation, real magnetic field is indistinguishable from the fictitious or statistical magnetic field introduced by a Chern-Simons term. Hence, real magnetic flux can be moved on to fermions and identified as statistical flux, so that the fermions get converted to anyons. The interesting result that was discovered in this approach was that precisely at those values of the magnetic field where the FQHE occured, all of the real magnetic flux could be transferred onto the fermions so that they could be transmuted to bosons. Pictorially, this procedure can be represented as shown in Fig.(25).

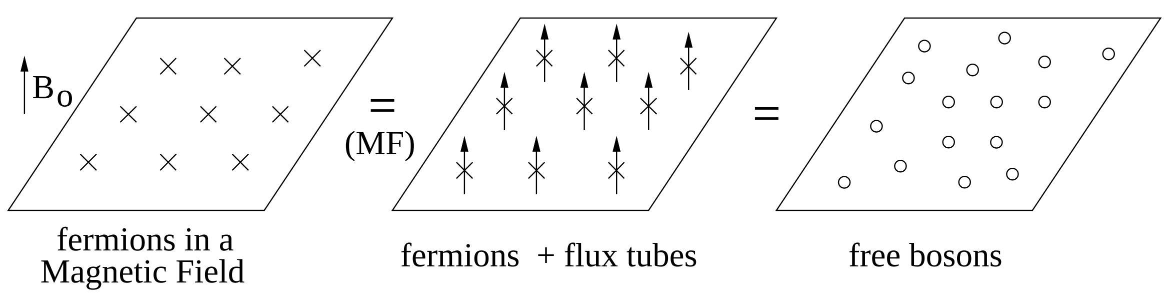


Fig. 25

An equivalent picture (which is the one that we shall implement formally) is to consider the fermions as bosons with attached flux tubes. Then the FQHE occurs precisely when the external magnetic field cancels (in a mean field sense) the effect of the flux tubes. Once again, this can be depicted as shown in Fig.(26).

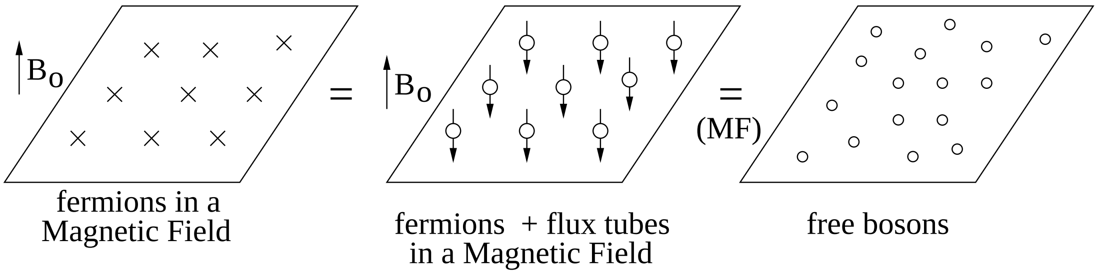


Fig. 26

In either picture, FQHE occurs when the fermions turn into free bosons on the average. Thus, the FQHE (or formation of an incompressible fluid) is equivalent to the formation of a Bose condensate. The stability of the FQHE states is 'explained' by the well-known fact that bosons can lower their energy by Bose condensing.

Let us now derive the above explanation more formally, starting from the microscopic Hamiltonian for electrons in an external electromagnetic potential given by

$$H = \frac{1}{2m}\sum_{i=1}^{N}(\mathbf{p}_i - e\mathbf{A}(\mathbf{r}_i))^2 + \sum_{i=1}^{N} eA_0(\mathbf{r}_i) + \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j), \tag{6.36}$$

where $V(\mathbf{r}_i - \mathbf{r}_j)$ is the repulsive interelectron potential. (In writing this Hamiltonian, we have dropped the background neutralising potential, which, of course, is always present.) The solution $\psi(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_N)$ to the Schrodinger equation

$$H\psi(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_N) = E\psi(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_N) \tag{6.37}$$

has to be antisymmetric under the exchange of any two coordinates, since the particles are fermions. These fermions can also be written as bosons interacting with an appropriate Chern-Simons gauge field. Hence, an equivalent formulation of the problem is given by

$$H' = \frac{1}{2m}\sum_{i=1}^{N}(\mathbf{p}_i - eA(\mathbf{r}_i) - e\mathbf{a}_i)^2 + \sum_{i=1}^{N} eA_0 + \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j), \tag{6.38}$$

with

$$H'\phi(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_N) = E\phi(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_N). \tag{6.39}$$

ϕ is now symmetric under the exchange of coordinates, and $\mathbf{a}_i$ in Eq.(6.38) is given by

$$\mathbf{a}_i = \frac{1}{e}\frac{\alpha}{\pi}\sum_{j\neq i}\frac{\hat{z}\times(\mathbf{r}_i-\mathbf{r}_j)}{|\mathbf{r}_i-\mathbf{r}_j|^2} = \frac{1}{e}\frac{\alpha}{\pi}\sum_{j\neq i}\nabla_i\theta_{ij}, \tag{6.40}$$

where θ_{ij} is the angle made by $(\mathbf{r}_i-\mathbf{r}_j)$ with an arbitrary axis.

To prove that the two systems are really equivalent, all we need to show is that the factor $\alpha/e\pi$ in Eq.(6.40) is precisely the factor needed to convert fermions to bosons. Consider a unitary transformation on the wavefunction and the Hamiltonian in Eq.(6.39) —*i.e.*,

$$\phi(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N) \longrightarrow \tilde{\phi}(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N) = U\phi(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N) = e^{-\frac{i\alpha}{\pi}\Sigma_{i<j}\theta_{ij}}\phi(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N) \tag{6.41}$$

and

$$H' \longrightarrow UH'U^{-1} = H. \tag{6.42}$$

The transformed Schrodinger equation is given by

$$\begin{aligned} UH'U^{-1}U\phi(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N) &= EU\phi(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N) \\ \Rightarrow H\tilde{\phi}(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N) &= E\tilde{\phi}(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N). \end{aligned} \tag{6.43}$$

Furthermore, since θ_{ij} is the angle made by $(\mathbf{r}_i-\mathbf{r}_j)$ with an arbitrary axis,

$$\theta_{ij} = \theta_{ji} + \pi. \tag{6.44}$$

Hence, if ϕ is symmetric under exchange, $\tilde{\phi}$ picks up the phase $e^{i\alpha}$ and is fermionic whenever $\alpha = (2k+1)\pi$. For these values of α, the systems described by Eqs.(6.36) and (6.37) on the one hand and Eqs.(6.38) and (6.39) on the other, are equivalent —*i.e.*, $\tilde{\phi}(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N) = \psi(\mathbf{r}_1,\mathbf{r}_2,...\mathbf{r}_N)$. This is not really surprising, since the unitary transformation in Eq.(6.41) is just the generalisation to many particles of the gauge transformation used in Sec.(2), to go from the fermion gauge to the anyon gauge. $\alpha = (2k+1)\pi$ is the value of the statistics parameter for fermions to become bosons.

We can also explicitly check that the Hamiltonian in Eq.(6.38) can be obtained from a Chern-Simons Lagrangian. From Eq.(6.40), we can show that

$$b(\mathbf{r}_i) = \nabla \times \mathbf{a} = \frac{2\alpha}{e} \sum_{j \neq i} \delta(\mathbf{r}_i - \mathbf{r}_j) = \frac{2\alpha\rho}{e} \tag{6.45}$$

where ρ is the density of particles. Comparing with Eq.(5.5), we see that this is precisely the form of the equations of motion derived from a Chern-Simons Lagrangian. Hence, the appropriate action that describes the effective field theory of the FQHE problem is given by

$$\begin{aligned}
& S = S_a + S_\phi \\
\text{where} \quad & S_a = \int d^3x \{ \frac{e^2}{4\alpha} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \} \\
\text{and} \quad & S_\phi = \int d^3x \{ \phi^\dagger [i\partial_t - e(A_0 + a_0)]\phi + \mu\phi^\dagger\phi + \frac{1}{2m}[\phi^\dagger(-i\nabla - e\mathbf{A} - e\mathbf{a})^2\phi] \} \\
& \quad - \frac{1}{2} \int d^2\mathbf{r}_1 d^2\mathbf{r}_2 \{ \phi^\dagger(\mathbf{r}_1)\phi^\dagger(\mathbf{r}_2) V(\mathbf{r}_1 - \mathbf{r}_2) \phi(\mathbf{r}_1)\phi(\mathbf{r}_2) \}.
\end{aligned} \tag{6.46}$$

This field theory formalism now enables us to look for minima of the action (which correspond to the usual mean field theories) and incorporate quantum corrections by expanding around the minima.

Let us first consider the case where there exists a non-zero magnetic field, but no electric field so that

$$A_0 = 0, \quad \text{and} \quad \epsilon^{ij}\partial_i A_j = -B_0 = \text{constant}. \tag{6.47}$$

Here, the action in Eq.(6.46) is minimised by choosing

$$\phi = \sqrt{\rho},\ \mathbf{A} = - < \mathbf{a} >, \quad \text{and} \quad a_0 = 0, \tag{6.48}$$

where the density of particles ρ is a constant. ($\phi^\dagger\phi = \rho$ is enforced by choosing the chemical potential μ suitably). Since the statistical magnetic field b is related to ρ (Eq.(6.45)) and Eq.(6.48) relates b to the external magnetic field B_0, this minimisation is only possible

when

$$B_0 = b = \frac{2\alpha\rho}{e} \tag{6.49}$$

or when

$$\nu = \frac{\rho}{\rho_B} = \frac{\rho}{eB_0/2\pi} = \frac{\pi}{\alpha} = \frac{1}{(2k+1)} \tag{6.50}$$

where $\rho_B = eB/2\pi =$ degeneracy of each Landau level and we have substituted $\alpha = (2k+1)\pi$. Thus, the action is minimised at the densities for which the filling fraction $\nu = 1/$odd integer -*i.e.*, the Laughlin fractions. To prove that the vacua at these fractions are incompressible, we also need to show that all excitations over these vacua are massive. For the quasiparticle excitations, we have already seen in Sec.(5) that the CSLG theory or the abelian Higgs model with a CS term has charged vortex solutions that have finite (non-zero) energies and fractional spin. Collective excitations, which are fluctuations of $(\mathbf{A} + \mathbf{a})$ (which could be massless, in principle, as for anyon superconductivity) are also massive because of the spontaneous breakdown of the U(1) symmetry caused by the vacuum expectation value of ϕ. Hence, it appears reasonable to identify the classical minima of the effective field theory with the Laughlin states.

Let us now calculate the Hall conductance by applying an external electric field $E_i = -\partial_i A_0$ along with the external magnetic field $B_0 = -\epsilon^{ij}\partial_i A_j$. The observed current can be obtained from the action by using

$$j_i = \frac{\partial S}{\partial A_i} = \frac{\partial S_\phi}{\partial A_i} = \frac{\partial S_\phi}{\partial a_i} \tag{6.51}$$

since A_i and a_i enter S_ϕ symmetrically. The equations of motion with respect to a_i are given by

$$\frac{\partial S}{\partial a_i} = 0 \Rightarrow \frac{\partial S_a}{\partial a_i} + \frac{\partial S_\phi}{\partial a_i} = 0. \tag{6.52}$$

Substituting for $\partial S_\phi/\partial a_i$ in Eq.(6.51), we see that

$$j_i = -\frac{\partial S_a}{\partial a_i} = \frac{e^2}{2\alpha}\epsilon^{0ij}(\partial_0 a_j - \partial_j a_0). \tag{6.53}$$

Now, from the minimum energy ansatz in Eq.(6.48) for a static magnetic field and the

ansatz $E_i = \partial_i A_0$ for the external electric field, we have

$$\partial_0 a_j = -\partial_0 A_j = 0 \quad \text{and} \quad \partial_j a_0 = -\partial_j A_0 = E_j. \tag{6.54}$$

Thus, the current can be written as

$$j_i = \frac{e^2}{2\pi(2k+1)}\epsilon^{ij} E_j, \tag{6.55}$$

so that we obtain the Hall conductance as

$$\sigma_{xy} = \frac{e^2}{2\pi(2k+1)} \tag{6.56}$$

which agrees with the semiclassical answer for the Hall conductivity $\sigma_{xy} = e\rho/B$ given in Eq.(6.5), since $\rho/B = e\nu/2\pi$ and $\nu = 1/(2k+1)$. Hence, the LGCS theory 'explains' the stability of the FQHE states at the Laughlin fractions and gives the right Hall conductivity at these densities.

The charm of the LGCS theory lies in the fact that besides reproducing the phenomenology of the FQHE, it also provides a formalism for addressing questions like the existence of an order parameter and off-diagonal long range order in the system. The full implications of the LGCS theory are yet to be understood, although the theory has been taken considerably further [46]. This is an active field of research and many more questions remain, both to be formulated and answered.

Problems

1. One of the points which was glossed over in the lecture was the question of how states get 'pinned' or 'localised' by impurities. Just to get a feel for this point, solve the quantum mechanical problem of a single fermion in a magnetic field and in the presence of an impurity scattering potential $V_I = \lambda\delta(x - x_0)\delta(y - y_0)$. (If you get stuck, look up Ref.[47].)

2. A subtlety.

   Fermions with attached flux tubes of strength $2k\phi_0$ remain fermions because the

statistical parameter $\alpha = (\frac{e}{2})\phi = (\frac{e}{2})2k(\frac{2\pi}{e}) = 2k\pi$ is defined only modulo 2π. So why are 'composite fermions' different from ordinary fermions and lead to different physics (FQHE vs. IQHE)?

7. Conclusion

We conclude this review by pointing out that these lectures are merely meant to serve as an eye-opener to the ever-expanding field of anyon physics. Besides, the several 'known' unanswered questions in the field, there probably remain many more unexplored and unexpected connections between CS theories and other topological and non-topological field theories. Applications of anyon ideas to other phenomena in condensed matter physics also remain a distinct possibility. Our hope is to inspire many more readers to join the 'anyon bandwagon'.

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